

No Small Linear Program Approximates Vertex Cover within a Factor $2 - \varepsilon$

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Abstract

The vertex cover problem is one of the most important and intensively studied combinatorial optimization problems. Khot and Regev [32, 33] proved that the problem is NP-hard to approximate within a factor $2 - \varepsilon$, assuming the Unique Games Conjecture (UGC). This is tight because the problem has an easy 2-approximation algorithm. Without resorting to the UGC, the best inapproximability result for the problem is due to Dinur and Safra [17, 18]: vertex cover is NP-hard to approximate within a factor 1.3606.

We prove the following unconditional result about linear programming (LP) relaxations of the problem: every LP relaxation that approximates vertex cover within a factor $2 - \varepsilon$ has super-polynomially many inequalities. As a direct consequence of our methods, we also establish that LP relaxations (as well as SDP relaxations) that approximate the independent set problem within any constant factor have super-polynomial size.

Keywords: Extended formulations, Hardness of approximation, Independent set, Linear programming, Vertex cover.

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1 Introduction

In this paper we prove tight inapproximability results for VERTEX COVER with respect to linear programming relaxations of polynomial size. VERTEX COVER is the following classic problem: given a graph $G = (V, E)$ together with vertex costs $c_v \geq 0$, $v \in V$, find a minimum cost set of vertices $U \subseteq V$ such that every edge has at least one endpoint in U . Such a set of vertices meeting every edge is called a *vertex cover*.

It is well known that the LP relaxation

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall uv \in E \\ & 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned} \tag{1.1}$$

approximates VERTEX COVER within a factor 2. (See e.g., Hochbaum [27] and the references therein.) This means that for every cost vector there exists a vertex cover whose cost is at most 2 times the optimum value of the LP. In fact, the (global) *integrality gap* of this LP relaxation, the worst-case ratio over all graphs and all cost vectors between the minimum cost of an integer solution and the minimum cost of a fractional solution, equals 2.

One way to make the LP relaxation (1.1) stronger is by adding valid inequalities. Here, a *valid inequality* is a linear inequality $\sum_{v \in V} a_v x_v \geq \beta$ that is satisfied by every integral solution. Adding all possible valid inequalities to (1.1) would clearly decrease the integrality gap all the way from 2 to 1, and thus provide a perfect LP formulation. However, this would also yield an LP that we would not be able to write down or solve efficiently. Hence, it is necessary to restrict to more tangible families of valid inequalities.

For instance, if $C \subseteq V$ is the vertex set of an odd cycle in G , then $\sum_{v \in C} x_v \geq \frac{|C|+1}{2}$ is a valid inequality for vertex covers, known as an *odd cycle inequality*. However, the integrality gap remains 2 after adding all such inequalities to (1.1). More classes of inequalities are known beyond the odd cycle inequalities. However, we do not know any such class of valid inequalities that would decrease the integrality gap strictly below 2.

There has also been a lot of success ruling out concrete polynomial-size linear programming formulations arising from, e.g., the addition of a polynomial number of inequalities with sparse support or those arising from hierarchies, where new valid inequalities are generated in a systematic way. For instance, what about adding all valid inequalities supported on at most $o(n)$ vertices (where n denotes the number of vertices of G), or all those obtained by performing a few rounds of the Lovász-Schrijver (LS) lift-and-project procedure [39]? In their influential paper Arora, Bollobás and Lovász [2] (the journal version [3] is joint work with Tourlakis) proved that none of these broad classes of valid inequalities are sufficient to decrease the integrality gap to $2 - \varepsilon$ for any $\varepsilon > 0$.

The paper of Arora *et al.* was followed by many papers deriving stronger and stronger tradeoffs between number of rounds and integrality gap for VERTEX COVER and many other problems in various hierarchies, see the related work section below. The focus of this paper is to prove lower bounds in a more general model. Specifically, our goal is to understand the strength of *any* polynomial-size linear programming relaxation of VERTEX COVER independently of any hierarchy and irrespective of any complexity-theoretic assumption such as e.g., $P \neq NP$.

We will rule out *all possible* polynomial-size LP relaxations obtained from adding an *arbitrary* set of valid inequalities of polynomial size. By “all possible LP relaxations”, we mean that the variables of the LP can be chosen arbitrarily. They do not have to be the vertex-variables of (1.1).

Contribution

We consider the general model of LP relaxations as in [13], see also [10]. Given an n -vertex graph $G = (V, E)$, a system of linear inequalities $Ax \geq b$ in \mathbb{R}^d , where $d \in \mathbb{N}$ is arbitrary, defines an *LP relaxation* of VERTEX COVER (on G) if the following conditions hold:

Feasibility: For every vertex cover $U \subseteq V$, we have a feasible vector $x^U \in \mathbb{R}^d$ satisfying $Ax^U \geq b$.

Linear objective: For every vertex-costs $c \in \mathbb{R}_+^V$, we have an affine function (degree-1 polynomial) $f_c : \mathbb{R}^d \rightarrow \mathbb{R}$.

Consistency: For all vertex covers $U \subseteq V$ and vertex-costs $c \in \mathbb{R}_+^V$, the condition $f_c(x^U) = \sum_{v \in U} c_v$ holds.

For every vertex-costs $c \in \mathbb{R}_+^V$, the LP $\min\{f_c(x) \mid Ax \geq b\}$ provides a guess on the minimum cost of a vertex cover. This guess is always a lower bound on the optimum.

We allow arbitrary computations for writing down the LP, and do not bound the size of the coefficients. We only care about the following two parameters and their relationship: the *size* of the LP relaxation, defined as the number of inequalities in $Ax \geq b$, and the (graph-specific) *integrality gap* which is the worst-case ratio over all vertex-costs between the true optimum and the guess provided by the LP, for this particular graph G and LP relaxation.

This framework subsumes the polyhedral-pair approach in extended formulations [8]; see also [43]. We refer the interested reader to the surveys [15, 28] for an introduction to extended formulations; see also Section 4 for more details.

In this paper, we prove the following result about LP relaxations of VERTEX COVER and, as a byproduct, INDEPENDENT SET.¹

Theorem 1.1. *For infinitely many values of n , there exists an n -vertex graph G such that: (i) Every size- $n^{o(\log n / \log \log n)}$ LP relaxation of VERTEX COVER on G has integrality gap $2 - o(1)$; (ii) Every size- $n^{o(\log n / \log \log n)}$ LP relaxation of INDEPENDENT SET on G has integrality gap $\omega(1)$.*

This solves an open problem that was posed both by Singh [51] and Chan, Lee, Raghavendra and Steurer [13]. In fact, Singh conjectured that every compact (that is, polynomial size), *symmetric* extended formulation for VERTEX COVER has integrality gap at least $2 - \varepsilon$. We prove that his conjecture holds, even if asymmetric extended formulations are allowed.²

Our result for the INDEPENDENT SET problem is even stronger than Theorem 1.1, as we are also able to rule out any polynomial size SDP with constant integrality gap for this problem. Furthermore, combining our proof strategy with more complex techniques we can prove a result similar to Theorem 1.1 for q -UNIFORM-VERTEX-COVER (that is, vertex cover in q -uniform *hypergraphs*), for any fixed $q \geq 2$. For that problem, every size $n^{o(\log n / \log \log n)}$ LP relaxation has integrality gap $q - o(1)$. This generalizes our result on (graph) VERTEX COVER.

In the general model of LP relaxations outlined above, the LPs are designed with the knowledge of the graph $G = (V, E)$; this is a *non-uniform* model as the LP can depend on the graph. It captures the natural LP relaxations for VERTEX COVER and INDEPENDENT SET whose constraints depend on the graph structure. This is in contrast to previous lower bound results ([8, 11, 9]) on the LP formulation complexity of INDEPENDENT SET, which are of a *uniform* nature: In those works, the

¹Recall that an *independent set* (stable set) in graph $G = (V, E)$ is a set of vertices $I \subseteq V$ such that no edge has both endpoints in I . INDEPENDENT SET is the corresponding maximization problem: given a graph together with a weight for each vertex, find a maximum weight independent set.

²Note that in some cases imposing symmetry is a severe restriction, see Kaibel, Pashkovich and Theis [29].

formulation of the LP relaxation was agnostic to the input graph and only allowed to depend on the number of vertices of the graph; see [10] for a discussion of uniformity vs. non-uniformity. In general non-uniform models are stronger (and so are lower bounds for it) and interestingly, this allows for stronger LP relaxations for INDEPENDENT SET than NP-hardness would predict. This phenomenon is related to the approximability of problems with preprocessing. In Section 5, we observe that a result of Feige and Jozeph [21] implies that there exists a size- $O(n)$ LP formulation for approximating INDEPENDENT SET within a multiplicative factor of $O(\sqrt{n})$.

Related work

Most of the work on extended formulations is ultimately rooted in Yannakakis’s famous paper [54, 55] in which he proved that every symmetric extended formulation of the matching polytope and (hence) TSP polytope of the n -vertex complete graph has size $2^{\Omega(n)}$. Yannakakis’s work was motivated by approaches to proving $P = NP$ by providing small (symmetric) LPs for the TSP, which he ruled out.

The paper of Arora *et al.* [2, 3] revived Yannakakis’s ideas in the context of hardness of approximation and provided lower bounds for VERTEX COVER in LS. It marked the starting point for a whole series of papers on approximations via hierarchies. Shortly after Arora *et al.* proved that performing $O(\log n)$ rounds of LS does not decrease the integrality gap below 2, Schoenebeck, Trevisan and Tzoulakis [48] proved that this also holds for $o(n)$ rounds of LS. A similar result holds for the stronger Sherali-Adams (SA) hierarchy [49]: Charikar, Makarychev and Makarychev [14] showed that $\Omega(n^\delta)$ rounds of SA are necessary to decrease the integrality gap beyond $2 - \varepsilon$ for some $\delta = \delta(\varepsilon) > 0$.

Beyond linear programming hierarchies, there are also semidefinite programming (SDP) hierarchies, e.g., Lovász-Schrijver (LS+) [39] and Sum-of-Squares/Lasserre [42, 35, 36]. Georgiou, Magen, Pitassi and Tzoulakis [24] proved that $O(\sqrt{\log n / \log \log n})$ rounds of LS+ does not approximate VERTEX COVER within a factor better than 2. In this paper, we focus mostly on the LP case.

Other papers in the “hierarchies” line of work include [16, 23, 47, 34, 44, 53, 30, 1, 6].

Although hierarchies are a powerful tool, they have their limitations. For instance, $o(n)$ rounds of SA does not give an approximation of KNAPSACK with a factor better than 2 [30]. However, for every $\varepsilon > 0$, there exists a size- $n^{1/\varepsilon + O(1)}$ LP relaxation that approximates KNAPSACK within a factor of $1 + \varepsilon$ [7].

Besides the study of hierarchy approaches, there was a distinct line of work inspired directly by Yannakakis’s paper that sought to study the power of general (linear) extended formulations, independently of any hierarchy, see e.g., [45, 22, 8, 6, 11, 9, 46]. Limitations of semidefinite extended formulations were also studied recently, see [12, 38].

The lines of work on hierarchies and (general) extended formulations in the case of CONSTRAINT SATISFACTION PROBLEMS (CSPs) were merged in the work of Chan *et al.* [13]. Their main result states that for Max-CSPs, SA is best possible among all LP relaxations in the sense that if there exists a size- n^r LP relaxation approximating a given Max-CSP within factor α then performing $2r$ rounds of SA would also provide a factor- α approximation. They obtained several strong LP inapproximability results for Max-CSPs such as MAX CUT and MAX 3-SAT. This result was recently strengthened in a breakthrough by Lee, Raghavendra, and Steurer [38], who obtained analogous results showing (informally) that the Sum-of-Squares/Lasserre hierarchy is best possible among all SDP relaxations for Max-CSPs.

Braun, Pokutta and Zink [10] developed a framework for proving size lower bounds on LP relaxations via reductions. Using [13] and FGLSS graphs [20], they obtained a $n^{\Omega(\log n / \log \log n)}$ size

lower bound for approximating VERTEX COVER within a factor of $1.5 - \varepsilon$ and INDEPENDENT SET within a factor of $2 - \varepsilon$. Our paper improves these inapproximability factors to a tight $2 - \varepsilon$ and any constant, respectively.

Outline

The framework in Braun *et al.* [10] formalizes sufficient properties of reductions for preserving inapproximability with respect to extended formulations / LP relaxations; this reduction mechanism does not capture all known reductions due to certain linearity and independence requirements. Using this framework, they gave a reduction from MAX CUT to VERTEX COVER yielding the aforementioned result.

A natural approach for strengthening the hardness factor is to reduce from UNIQUE GAMES instead of MAX CUT (since VERTEX COVER is known to be UNIQUE GAMES-hard to approximate within a factor $2 - \varepsilon$). However, one obstacle is that, in known reductions from UNIQUE GAMES, the optimal value of the obtained VERTEX COVER instance is not *linearly* related to the value of the UNIQUE GAMES instance. This makes these reductions unsuitable for the framework in [10] (see Definition 4.3).

We overcome this obstacle by designing a two-step reduction. In the first step (Section 3), we interpret the “one free bit” PCP test of Bansal and Khot [4] as a reduction from a UNIQUE GAMES instance to a “one free bit” CSP (1F-CSP). We then use the family of SA integrality gap instances for the UNIQUE GAMES problem constructed by Charikar *et al.* [14], to construct a similar family for this CSP. This, together with the main result of Chan *et al.* [13] applied to this particular CSP, implies that no size- $n^{o(\log n / \log \log n)}$ LP relaxation can provide a constant factor approximation for 1F-CSP. In the second step (Section 4), a reduction from 1F-CSP to VERTEX COVER, in the framework of Braun *et al.* [10], then yields our main result.

Finally, following a slightly different and more challenging route we prove tight hardness of approximation for LP relaxations of q -UNIFORM-VERTEX-COVER for every $q \geq 2$. This is done in Section 6.

2 Preliminaries

We shall now present required tools and background. In Sections 2.1 and 2.2 we define the class of CONSTRAINT SATISFACTION PROBLEMS and the Sherali-Adams (SA) hierarchy, respectively.

2.1 CONSTRAINT SATISFACTION PROBLEMS

The class of CONSTRAINT SATISFACTION PROBLEMS (CSPs) captures a large variety of combinatorial problems, like MAX CUT and MAX 3-SAT. In general, we are given a collection of *predicates* $\mathcal{P} = \{P_1, \dots, P_m\}$ (or *constraints* $C = \{C_1, \dots, C_m\}$) where each P_i is of the form $P_i : [R]^n \mapsto \{0, 1\}$, where $[R] := \{1, \dots, R\}$ is the *domain* and n is the number of variables. We will be mainly interested in the family of CSPs where each predicate P is associated with a set of distinct indices $S_P = \{i_1, \dots, i_k\} \subset [n]$ and is of constant arity k , i.e., $P : [R]^k \mapsto \{0, 1\}$. In this terminology, for $x \in [R]^n$ we set $P(x) := P(x_{i_1}, \dots, x_{i_k})$. The goal in such problems is to find an assignment for $x \in [R]^n$ in such a way as to maximize the total fraction of satisfied predicates.

The *value* of an assignment $x \in [R]^n$ for a CSP instance \mathcal{I} is defined as

$$\text{Val}_{\mathcal{I}}(x) := \frac{1}{m} \sum_{i=1}^m P_i(x) = \mathbb{E}_{P \in \mathcal{P}} [P(x)],$$

and the optimal value of such instance I , denoted by $\text{OPT}(I)$ is

$$\text{OPT}(I) = \max_{x \in [R]^n} \text{Val}_I(x).$$

Often, we will consider *binary* CSPs, that is, with domain size $R = 2$. Given a binary predicate $P : \{0, 1\}^k \mapsto \{0, 1\}$, the *free bit* complexity of P is defined to be $\log_2(|\{z \in \{0, 1\}^k : P(z) = 1\}|)$. For example the MAX CUT predicate $x_i \oplus x_j$ has a free bit complexity of *one*, since the only *two* accepting configurations are $(x_i = 0, x_j = 1)$ and $(x_i = 1, x_j = 0)$.

For the LP-hardness of VERTEX COVER and INDEPENDENT SET (i.e., Sections 3 and 4), we will be interested in a one free bit binary CSP, that we refer to as 1F-CSP, defined as follows:

Definition 2.1 (1F-CSP). A 1F-CSP instance of arity k is a binary CSP over a set of variables $\{x_1, \dots, x_n\}$ and a set of constraints $C = \{C_1, \dots, C_m\}$ such that each constraint $C \in C$ is of arity k and has only two accepting configurations out of the 2^k possible ones.

2.2 Sherali-Adams Hierarchy

We define the canonical relaxation for $\text{CONSTRAINT SATISFACTION PROBLEMS}$ as it is obtained by r -rounds of the Sherali-Adams (SA) hierarchy. We follow the notation as in e.g., [25]. For completeness we also describe in Appendix A why this relaxation is equivalent to the one obtained by applying the original definition of SA as a reformulation-linearization technique on a binary program.

Consider any CSP defined over n variables $x_1, \dots, x_n \in [R]$, with a set of m constraints $C = \{C_1, \dots, C_m\}$ where the arity of each constraint is at most k . Let $S_i = S_{C_i}$ denote the set of variables that C_i depends on. The r -rounds SA relaxation of this CSP has a variable $X_{(S, \alpha)}$ for each $S \subseteq [n], \alpha \in [R]^S$ with $|S| \leq r$. The intuition is that $X_{(S, \alpha)}$ models the indicator variable whether the variables in S are assigned the values in α . The r -rounds SA relaxation with $r \geq k$ is now

$$\begin{aligned} \max \quad & \frac{1}{m} \sum_{i=1}^m \sum_{\alpha \in [R]^{S_i}} C_i(\alpha) \cdot X_{(S_i, \alpha)} \\ \text{s.t.} \quad & \sum_{u \in [R]} X_{(S \cup \{j\}, \alpha \circ u)} = X_{(S, \alpha)} \quad \forall S \subseteq [n] : |S| < r, \alpha \in [R]^S, j \in [n] \setminus S, \\ & X_{(S, \alpha)} \geq 0 \quad \forall S \subseteq [n] : |S| \leq r, \alpha \in [R]^S, \\ & X_{(\emptyset, \emptyset)} = 1. \end{aligned} \tag{2.1}$$

Here we used the notation $(S \cup \{j\}, \alpha \circ u)$ to extend the assignment α to assign u to the variable indexed by j . Note that the first set of constraints say that the variables should indicate a consistent assignment.

Instead of dealing with the constraints of the Sherali-Adams LP relaxation directly, it is simpler to view each solution of the Sherali-Adams LP as a consistent collection of local distributions over partial assignments.

Suppose that for every set $S \subseteq [n]$ with $|S| \leq r$, we are given a local distribution $\mathcal{D}(S)$ over $[R]^S$. We say that these distributions are *consistent* if for all $S' \subseteq S \subseteq [n]$ with $|S| \leq r$, the marginal distribution induced on $[R]^{S'}$ by $\mathcal{D}(S)$ coincides with that of $\mathcal{D}(S')$.

The equivalence between SA solutions and consistent collections of local distributions basically follows from the definition of (2.1) and is also used in [14] and [13] that are most relevant to our approach. More specifically, we have

Lemma 2.2 (Lemma 1 in [25]). *If $\{\mathcal{D}(S)\}_{S \subseteq [n]: |S| \leq r}$ is a consistent collection of local distributions then*

$$X_{(S,\alpha)} = \mathbb{P}_{\mathcal{D}(S)}[\alpha]$$

is a feasible solution to (2.1).

Moreover, we have the other direction.

Lemma 2.3. *Consider a feasible solution $(X_{(S,\alpha)})_{S \subseteq [n]: |S| \leq r, \alpha \in [R]^S}$ to (2.1). For each $S \subseteq [n]$ with $|S| \leq r$, define*

$$\mathbb{P}_{\mathcal{D}(S)}[\alpha] = X_{(S,\alpha)} \quad \text{for each } \alpha \in [R]^S.$$

Then $(\mathcal{D}(S))_{S \subseteq [n]: |S| \leq r}$ forms a consistent collection of local distributions.

Proof. Note that, for each $S \subseteq [n]$ with $|S| \leq r$, $\mathcal{D}(S)$ is indeed a distribution because by the equality constraints of (2.1)

$$\sum_{\alpha \in [R]^S} \mathbb{P}_{\mathcal{D}(S)}[\alpha] = \sum_{\alpha \in [R]^S} X_{(S,\alpha)} = \sum_{\alpha' \in [R]^{S'}} X_{(S',\alpha')} = X_{(\emptyset,\emptyset)} = 1$$

where $S' \subseteq S$ is arbitrary; and moreover $\mathbb{P}_{\mathcal{D}(S)}[\alpha] = X_{(S,\alpha)} \geq 0$. Similarly we have, again by the equality constraints of (2.1), that for each $S' \subseteq S$ and $\alpha' \in [R]^{S'}$

$$\mathbb{P}_{\mathcal{D}(S')}[\alpha'] = X_{(S',\alpha')} = \sum_{\alpha'' \in [R]^{S \setminus S'}} X_{(S,\alpha' \circ \alpha'')} = \sum_{\alpha'' \in [R]^{S \setminus S'}} \mathbb{P}_{\mathcal{D}(S)}[\alpha' \circ \alpha'']$$

so the local distributions are consistent. \square

When a SA solution $(X_{(S,\alpha)})$ is viewed as consistent collection $\{\mathcal{D}(S)\}$ of local distributions, the value of the SA solution can be computed as

$$\frac{1}{m} \sum_{i=1}^m \sum_{\alpha \in [R]^{S_i}} C_i(\alpha) \cdot X_{(S_i,\alpha)} = \mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \mathcal{D}(S_C)}[\alpha \text{ satisfies } C] \right]$$

where S_C is the support of constraint C .

3 Sherali-Adams Integrality Gap for 1F-CSP

In this section we establish Sherali-Adams integrality gaps for 1F-CSP and by virtue of [13] this extends to general LPs. The proof uses the idea of [14] to perform a reduction between problems that preserves the Sherali-Adams integrality gap.

Specifically, we show that the reduction by Bansal and Khot [4] from the UNIQUE GAMES problem to 1F-CSP also provides a large Sherali-Adams integrality gap for 1F-CSP, assuming that we start with a Sherali-Adams integrality gap instance of UNIQUE GAMES. As large Sherali-Adams integrality gap instances of UNIQUE GAMES were given in [14], this implies the aforementioned integrality gap of 1F-CSP.

3.1 UNIQUE GAMES

The UNIQUE GAMES problem is defined as follows:

Definition 3.1. A UNIQUE GAMES instance $\mathcal{U} = (G, [R], \Pi)$ is defined by a graph $G = (V, E)$ over a vertex set V and edge set E , where every edge $uv \in E$ is associated with a bijection map $\pi_{u,v} \in \Pi$ such that $\pi_{u,v} : [R] \mapsto [R]$ (we set $\pi_{v,u} := \pi_{u,v}^{-1}$). Here, $[R]$ is known as the label set. The goal is to find a labeling $\Lambda : V \mapsto [R]$ that maximizes the number of satisfied edges, where an edge uv is satisfied by Λ if $\pi_{u,v}(\Lambda(u)) = \Lambda(v)$.

The following very influential conjecture, known as the UNIQUE GAMES conjecture, is due to Khot [31].

Conjecture 3.2. *For any $\zeta, \delta > 0$, there exists a sufficiently large constant $R = R(\zeta, \delta)$ such that the following promise problem is NP-hard. Given a UNIQUE GAMES instance $\mathcal{U} = (G, [R], \Pi)$, distinguish between the following two cases:*

1. *Completeness: There exists a labeling Λ that satisfies at least $(1 - \zeta)$ -fraction of the edges.*
2. *Soundness: No labeling satisfies more than δ -fraction of the edges.*

We remark that the above conjecture has several equivalent formulations via fairly standard transformations. In particular, one can assume that the graph G is bipartite and regular [32].

The starting point of our reduction is the following Sherali-Adams integrality gap instances for the UNIQUE GAMES problem. Note that UNIQUE GAMES are CONSTRAINT SATISFACTION PROBLEMS and hence here and in the following, we are concerned with the standard application of the Sherali-Adams hierarchy to CSPs.

Theorem 3.3 ([14]). *Fix a label size $R = 2^\ell$, a real $\delta \in (0, 1)$ and let $\Delta := \lceil C(R/\delta)^2 \rceil$ (for a sufficiently large constant C). Then for every positive ε there exists $\kappa > 0$ depending on ε such that for infinitely many n there exists an instance of UNIQUE GAMES on a Δ -regular n -vertex graph $G = (V, E)$ so that:*

1. *The value of the optimal solution is at most $\frac{1}{R} \cdot (1 + \delta)$.*
2. *There exists a solution to the LP relaxation obtained after $r = n^\kappa$ rounds of the Sherali-Adams relaxation of value $1 - \varepsilon$.*

3.2 Reduction from UNIQUE GAMES to 1F-CSP

We first describe the reduction from UNIQUE GAMES to 1F-CSP that follows the construction in [4]. We then show that it also preserves the Sherali-Adams integrality gap.

Reduction. Let $\mathcal{U} = (G, [R], \Pi)$ be a UNIQUE GAMES instance over a regular bipartite graph $G = (V, W, E)$. Given \mathcal{U} , we construct an instance \mathcal{I} of 1F-CSP. The reduction has two parameters: $\delta > 0$ and $\varepsilon > 0$, where ε is chosen such that εR is an integer (taking $\varepsilon = 2^{-q}$ for some integer $q \geq 0$ guarantees this). We then select t to be a large integer depending on ε and δ .

The resulting 1F-CSP instance \mathcal{I} will be defined over $2^R |W|$ variables and $c|V|$ constraints, where $c := c(R, \varepsilon, t, \Delta)$ is a function of the degree Δ of the UNIQUE GAMES instance, and the constants R, t and ε .³ For our purposes, the UNIQUE GAMES integrality gap instance that we start from has constant degree Δ , and hence c is a constant.

³More precisely $c(R, \varepsilon, t, \Delta)$ is exponential in the constants R, t and ε , and polynomial in Δ

Before we proceed, we stress the fact that our reduction is essentially the same as the one free bit test $F_{\varepsilon,t}$ in [4], but casted in the language of CONSTRAINT SATISFACTION PROBLEMS. The test $F_{\varepsilon,t}$ expects a labeling $\Lambda : W \mapsto [R]$ for the vertices of the UNIQUE GAMES instance, where each label $\Lambda(w) \in [R]$ is encoded using a 2^R bit string. To check the validity of this labeling, the verifier picks a vertex $v \in V$ uniformly at random, and a sequence of t neighbors w_1, \dots, w_t of v randomly and independently from the neighborhood of v , and asks the provers about the labels of $\{w_1, \dots, w_t\}$ under the labeling Λ . It then accepts if the answers of the provers were convincing, i.e., the labels assigned to $\{w_1, \dots, w_t\}$ satisfy the edges vw_1, \dots, vw_t simultaneously under $\pi_{v,w_1}, \dots, \pi_{v,w_t}$ respectively.

Instead of reading all of the $t2^R$ bits corresponding to the t labels, the verifier only reads a *random subset* of roughly $t2^{\varepsilon R}$ bits and is able to accept with high probability if the labeling was correct, and to reject with high probability if it was not correct. In our reduction, the variables of the 1F-CSP instance \mathcal{I} corresponds to the 2^R bits encoding the labels of each vertex of the UNIQUE GAMES instance we start from⁴, and the constraints corresponds to all possible tests that the verifier might perform according to the random choice of v , the random neighbors w_1, \dots, w_t and the random subset of bits read by the verifier. Instead of actually enumerating all possible constraints, we give a distribution of constraints which is the same as the distribution over the test predicates of $F_{\varepsilon,t}$.

We refer to the variables of \mathcal{I} as follows: it has a binary variable $\langle w, x \rangle$ for each $w \in W$ and $x \in \{0, 1\}^R$.⁵ For further reference, we let $\text{Var}(\mathcal{I})$ denote the set of variables of \mathcal{I} . The constraints of \mathcal{I} are picked according to the distribution in Figure 1.

It is crucial to observe that our distribution over the constraints exploits the locality of a UNIQUE GAMES solution. To see this, assume we performed the first two steps of Figure 1 and have thus far fixed a vertex $v \in V$ and t neighbors w_1, \dots, w_t , and let C_{v,w_1,\dots,w_t} denote the set of all possible constraints resulting from steps 3-4 (i.e., for all possible $x \in \{0, 1\}^R$ and $S \subseteq [R]$ of size εR). We will argue that if there exists a local assignment of labels for $\{v, w_1, \dots, w_t\}$ that satisfies the edges vw_1, \dots, vw_t , then we can derive a local assignment for the variables $\{\langle w, x \rangle : w \in \{w_1, \dots, w_t\} \text{ and } x \in \{0, 1\}^R\}$ that satisfies at least $1 - \varepsilon$ fraction of the constraints in C_{v,w_1,\dots,w_t} . This essentially follows from the completeness analysis of [4], and is formalized in Claim 3.6. This allows us to convert a *good* Sherali-Adams solution of the starting UNIQUE GAMES \mathcal{U} , to a *good* Sherali-Adams solution of the resulting 1F-CSP Instance \mathcal{I} . Moreover, in order to show that \mathcal{I} is a Sherali-Adams integrality gap instance for the 1F-CSP problem, we need to show that $\text{OPT}(\mathcal{I})$ is *small*. This follows from the soundness analysis of [4], where it was shown that:

Lemma 3.4 (soundness). *For any $\varepsilon, \eta > 0$ there exists an integer t so that $\text{OPT}(\mathcal{I}) \leq \eta$ if $\text{OPT}(\mathcal{U}) \leq \delta$ where $\delta > 0$ is a constant that only depends on ε, η and t .*

The above says that if we start with a UNIQUE GAMES instance \mathcal{U} with a small optimum then we also get a 1F-CSP instance \mathcal{I} of small optimum (assuming that the parameters of the reduction are set correctly). In [4], Bansal and Khot also proved the following completeness: if $\text{OPT}(\mathcal{U}) \geq 1 - \zeta$, then $\text{OPT}(\mathcal{I}) \geq 1 - \zeta t - \varepsilon$. However, we need the stronger statement: if \mathcal{U} has a Sherali-Adams solution of large value, then so does \mathcal{I} . The following lemma states this more formally, showing that we can transform a SA solution to the UNIQUE GAMES instance \mathcal{U} into a SA solution to the 1F-CSP instance \mathcal{I} of roughly the same value.

⁴For the reader familiar with hardness of approximation and PCP based hardness, we are using the long code to encode labels, so that each of these 2^R bits gives the value of the dictator function f evaluated on a different binary string $x \in \{0, 1\}^R$; for a valid encoding we have $f(x) = x_\ell$ where ℓ is the label that is encoded.

⁵ $\langle w, x \rangle$ should be interpreted as the long-code for $\Lambda(w)$ evaluated at $x \in \{0, 1\}^R$.

1. Pick a vertex $v \in V$ uniformly at random.
2. Pick t vertices w_1, \dots, w_t randomly and independently from the neighborhood $N(v) = \{w \in W : vw \in E\}$.
3. Pick $x \in \{0, 1\}^R$ at random.
4. Let $m = \varepsilon R$. Pick indices i_1, \dots, i_m randomly and independently from $[R]$ and let $S = \{i_1, \dots, i_m\}$ be the set of those indices.
5. Define the *sub-cubes*:

$$C_{x,S} = \{z \in \{0, 1\}^R : z_j = x_j \ \forall j \notin S\}$$

$$C_{\bar{x},S} = \{z \in \{0, 1\}^R : z_j = \bar{x}_j \ \forall j \notin S\}$$

6. Output the constraint on the variables $\{\langle w_i, z \rangle \mid i \in [t], \pi_{v,w_i}^{-1}(z) \in C_{x,S} \cup C_{\bar{x},S}\}$ that is true if for some bit $b \in \{0, 1\}$ we have

$$\begin{aligned} \langle w_i, z \rangle &= b && \text{for all } i \in [t] \text{ and } \pi_{v,w_i}^{-1}(z) \in C_{x,S}, \text{ and} \\ \langle w_i, z \rangle &= b \oplus 1 && \text{for all } i \in [t] \text{ and } \pi_{v,w_i}^{-1}(z) \in C_{\bar{x},S} \end{aligned}$$

where $\pi(z)$ for $z \in \{0, 1\}^R$ is defined as $\pi(z) := (z_{\pi(1)}, z_{\pi(2)}, \dots, z_{\pi(R)})$, and π^{-1} is the inverse map, i.e., $\pi^{-1}(z) \in C_{x,S}$ is equivalent to saying that there exists $y \in C_{x,S}$ such that $\pi(y) = z$.

Figure 1: Distribution for the 1F-CSP constraints

Lemma 3.5. *Let $\{\mu(S) \mid S \subseteq V \cup W, |S| \leq r\}$ be a consistent collection of local distributions defining a solution to the r -rounds Sherali-Adams relaxation of the regular bipartite UNIQUE GAMES instance \mathcal{U} . Then we can define a consistent collection of local distributions $\{\sigma(S) \mid S \subseteq \text{Var}(\mathcal{I}), |S| \leq r\}$ defining a solution to the r -rounds Sherali-Adams relaxation of the 1F-CSP instance \mathcal{I} so that*

$$\mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \sigma(S_C)} [\alpha \text{ satisfies } C] \right] \geq (1 - \varepsilon) \left(1 - t \cdot \mathbb{E}_{vw \in E} \left[\mathbb{P}_{(\Lambda(v), \Lambda(w) \sim \mu(\{v,w\}))} [\Lambda(v) \neq \pi_{w,v}(\Lambda(w))] \right] \right),$$

where t and ε are the parameters of the reduction, and $\sigma(S_C)$ is the distribution over the set of variables in the support S_C of constraint C .

We remark that the above lemma says that we can transform a SA solution to the UNIQUE GAMES instance \mathcal{U} of value close to 1, into a SA solution to the 1F-CSP instance \mathcal{I} of value also close to 1.

Proof of Lemma 3.5. Let $\{\mu(S) \mid S \subseteq V \cup W, |S| \leq r\}$ be a solution to the r -rounds SA relaxation of the UNIQUE GAMES instance \mathcal{U} , and recall that \mathcal{I} is the 1F-CSP instance obtained from applying the reduction. We will now use the collection of consistent local distributions of the UNIQUE GAMES instance, to construct another collection of consistent local distributions for the variables in $\text{Var}(\mathcal{I})$.

For every set $S \subseteq \text{Var}(\mathcal{I})$ such that $|S| \leq r$, let $T_S \subseteq W$ be the subset of vertices in the UNIQUE GAMES instance defined as follows:

$$T_S = \{w \in W : \langle w, x \rangle \in S\}. \quad (3.1)$$

We construct $\sigma(S)$ from $\mu(T_S)$ in the following manner. Given a labeling Λ_{T_S} for the vertices in T_S drawn from $\mu(T_S)$, define an assignment α_S for the variables in S as follows: for a variable

$\langle w, x \rangle \in S$, let $\ell = \Lambda_{T_S}(w)$ be the label of w according to Λ_{T_S} . Then the new assignment α_S sets $\alpha_S(\langle w, x \rangle) := x_\ell$.⁶ The aforementioned procedure defines a family $\{\sigma(S)\}_{S \subseteq \text{Var}(\mathcal{I}), |S| \leq r}$ of local distributions for the variables of the 1F-CSP instance \mathcal{I} .

To check that these local distributions are consistent, take any $S' \subseteq S \subseteq \text{Var}(\mathcal{I})$ with $|S| \leq r$, and denote by $T_{S'} \subseteq T_S$ their corresponding set of vertices as in (3.1). We know that $\mu(T_S)$ and $\mu(T_{S'})$ agree on $T_{S'}$ since the distributions $\{\mu(S)\}$ defines a feasible Sherali-Adams solution for \mathcal{U} , and hence by our construction, the local distributions $\sigma(S)$ and $\sigma(S')$ agree on S' . Combining all of these together, we get that $\{\sigma(S) \mid S \subseteq \text{Var}(\mathcal{I}), |S| \leq r\}$ defines a feasible solution for the r -round Sherali-Adams relaxation of the 1F-CSP instance \mathcal{I} .

It remains to bound the value of this feasible solution, i.e.,

$$\mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \sigma(S_C)} [\alpha \text{ satisfies } C] \right]. \quad (3.2)$$

In what follows, we denote by $\psi(\cdot)$ the operator mapping a labeling of the vertices in T_S to an assignment for the variables in S , i.e., $\psi(\Lambda_{T_S}) = \alpha_S$.

First note that a constraint $C \in \mathcal{C}$ of the 1F-CSP instance \mathcal{I} is defined by the choice of the vertex $v \in V$, the sequence of t neighbors $\mathcal{W}_v = (w_1, \dots, w_t)$, the random $x \in \{0, 1\}^R$, and the random set $S \subseteq [R]$ of size εR . We refer to such a constraint C as $C(v, \mathcal{W}_v, x, S)$. Thus we can rewrite (3.2) as

$$\mathbb{E}_{v, w_1, \dots, w_t} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\}), x, S} [\psi(\Lambda) \text{ satisfies } C(v, \mathcal{W}_v, x, S)] \right]. \quad (3.3)$$

Recall that the assignment $\psi(\Lambda)$ for the variables $\{\langle w, z \rangle : w \in \mathcal{W}_v \text{ and } z \in \{0, 1\}^R\}$ is derived from the labeling of the vertices in \mathcal{W}_v according to Λ . It was shown in [4] that if Λ satisfies the edges vw_1, \dots, vw_t simultaneously, then $\psi(\Lambda)$ satisfies $C(v, \mathcal{W}_v, x, S)$ with *high probability*. This is formalized in Claim 3.6, whose proof appears in Appendix B.

Claim 3.6. If Λ satisfies vw_1, \dots, vw_t simultaneously, then $\psi(\Lambda)$ satisfies $C(v, \mathcal{W}_v, x, S)$ with probability at least $1 - \varepsilon$. Moreover, if we *additionally* have that $\Lambda(v) \notin S$, then $\psi(\Lambda)$ always satisfies $C(v, \mathcal{W}_v, x, S)$.

It now follows from Claim 3.6 that for the assignment $\psi(\Lambda)$ to satisfy the constraint $C(v, \mathcal{W}_v, x, S)$, it is sufficient that the following two conditions hold *simultaneously*:

1. the labeling Λ satisfies the edges vw_1, \dots, vw_t ;
2. the label of v according to Λ lies outside the set S .

Equipped with this, we can use conditioning to lower-bound the probability inside the expectation in (3.3) by a product of two probabilities, where the first is

$$\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\}), x, S} [\psi(\Lambda) \text{ satisfies } C(v, \mathcal{W}_v, x, S) \mid \Lambda \text{ satisfies } vw_1, \dots, vw_t] \quad (3.4)$$

and the second is

$$\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\})} [\Lambda \text{ satisfies } vw_1, \dots, vw_t].$$

⁶Because $\langle w, x \rangle$ is supposed to be the dictator function of the ℓ th coordinate evaluated at x , this is only the correct way to set the bit $\langle w, x \rangle$.

Thus using Claim 3.6, we get

$$\begin{aligned} \mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \sigma(S_C)} [\alpha \text{ satisfies } C] \right] &\geq (1 - \varepsilon) \cdot \mathbb{E}_{v, w_1, \dots, w_t} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\})} [\Lambda \text{ satisfies } vw_1, \dots, vw_t] \right] \\ &\geq (1 - \varepsilon) \left(1 - \sum_{i=1}^t \mathbb{E}_{v, w_1, \dots, w_t} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\})} [\Lambda \text{ does not satisfy } vw_i] \right] \right) \end{aligned} \quad (3.5)$$

$$= (1 - \varepsilon) \left(1 - \sum_{i=1}^t \mathbb{E}_{v, w_1, \dots, w_t} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w_i\})} [\Lambda \text{ does not satisfy } vw_i] \right] \right) \quad (3.6)$$

$$= (1 - \varepsilon) \cdot \left(1 - t \cdot \mathbb{E}_{v, w} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w\})} [\Lambda \text{ does not satisfy } vw] \right] \right) \quad (3.7)$$

where (3.5) follows from the union bound, and (3.6) is due to the fact that the local distributions of the UNIQUE GAMES labeling are consistent, and hence agree on $\{v, w_i\}$. Note that the only difference between what we have proved thus far and the statement of the lemma, is that the expectation in (3.7) is taken over a random vertex v and a random vertex $w \in N(v)$, and not random edges. However, our UNIQUE GAMES instance we start from is regular, so picking a vertex v at random and then a random neighbor $w \in N(v)$, is equivalent to picking an edge at random from E . This concludes the proof. \square

Combining Theorem 3.3 with Lemmata 3.4 and 3.5, we get the following Corollary.

Corollary 3.7. *For every $\varepsilon, \eta > 0$, there exist an arity k and a real $\kappa > 0$ depending on ε and η such that for infinitely many n there exists an instance of 1F-CSP of arity k over n variables, so that*

1. *The value of the optimal solution is at most η .*
2. *There exists a solution to the LP relaxation obtained after $r = n^\kappa$ rounds of the Sherali-Adams relaxation of value at least $1 - \varepsilon$.*

Proof. Let $\mathcal{U} = (G, [R], \Pi)$ be a Δ -regular UNIQUE GAMES instance of Theorem 3.3 that is $\delta/4$ -satisfied with an $n_G^{2\kappa}$ -rounds Sherali-Adams solution of value $1 - \zeta$, where n_G is the number of vertices in G . Note that $G = (V, E)$ is not necessarily bipartite, and our starting instance of the reduction is bipartite. To circumvent this obstacle, we construct a new bipartite UNIQUE GAMES instance \mathcal{U}' from \mathcal{U} that is δ -satisfied with a Sherali-Adams solution of the same value, i.e., $1 - \zeta$. We will later use this new instance to construct our 1F-CSP instance over n variables that satisfies the properties in the statement of the corollary.

In what follows we think of δ, ζ and R as functions of ε and η , and hence fixing the latter two parameters enables us to fix the constant t of Lemma 3.4, and the constant degree Δ of Theorem 3.3. The aforementioned parameters are then sufficient to provide us with the constant arity k of the 1F-CSP instance, along with the number of its corresponding variables and constraints, that is linear in n_G .

We now construct the new UNIQUE GAMES instance \mathcal{U}' over a graph $G' = (V_1, V_2, E')$ and the label set $[R]$ from \mathcal{U} in the following manner:

- Each vertex $v \in V$ in the original graph is represented by two vertices v_1, v_2 , such that $v_1 \in V_1$ and $v_2 \in V_2$.
- Each edge $e = uv \in E$ is represented by two edges $e_1 = u_1v_2$ and $e_2 = u_2v_1$ in E' . The bijection maps π_{u_1, v_2} and π_{u_2, v_1} are the same as $\pi_{u, v}$.

Note that G' is bipartite by construction, and since G is Δ -regular, we get that G' is also Δ -regular.

We claim that no labeling $\Lambda' : V_1 \cup V_2 \mapsto [R]$ can satisfy more than δ fraction of the edges in \mathcal{U}' . Indeed, assume towards contradiction that there exists a labeling $\Lambda' : V_1 \cup V_2 \mapsto [R]$ that satisfies at least δ fraction of the edges. We will derive a labeling $\Lambda : V \mapsto [R]$ that satisfies at least $\delta/4$ fraction of the edges in \mathcal{U} as follows:

For every vertex $v \in V$, let $v_1 \in V_1$ and $v_2 \in V_2$ be its representative vertices in G' . Define $\Lambda(v)$ to be either $\Lambda'(v_1)$ or $\Lambda'(v_2)$ with equal probability.

Assume that at least one edge of $e_1 = u_1v_2$ and $e_2 = u_2v_1$ is satisfied by Λ' , then the edge $e = uv \in E$ is satisfied with probability at least $1/8$, and hence the expected fraction of satisfied edges in \mathcal{U} by Λ is at least $\delta/4$.

Moreover, we can extend the r -rounds Sherali-Adams solution of $\mathcal{U} \{\mathcal{D}(S)\}_{S \subseteq V; |S| \leq r}$, to a r -rounds Sherali-Adams solution $\{\mathcal{D}'(S)\}_{S \subseteq V_1 \cup V_2; |S| \leq r}$ for \mathcal{U}' with the same value. This can be done as follows: For every set $S = S_1 \cup S_2 \subseteq V_1 \cup V_2$ of size at most r , let $S_{\mathcal{U}} \subseteq V$ be the set of their corresponding vertices in G and define the local distribution $\mathcal{D}'(S)$ by mimicking the local distribution $\mathcal{D}(S_{\mathcal{U}})$, repeating labels if the same vertex $v \in S_{\mathcal{U}}$ has its two copies v_1 and v_2 in S .

Now let \mathcal{I} be the 1F-CSP instance over n variables obtained by our reduction from the UNIQUE GAMES instance \mathcal{U}' , where $n = 2^R n_G$. Since $\text{OPT}(\mathcal{U}') \leq \delta$, we get from Lemma 3.4 that $\text{OPT}(\mathcal{I}) \leq \eta$. Similarly, we know from Lemma 3.5 that using an $n_G^{2\kappa}$ -rounds Sherali-Adams solution for \mathcal{U}' , we can define an n^κ -rounds Sherali-Adams solution of \mathcal{I} of roughly the same value, where we used the fact that R is a constant and hence $(2^{-2R\kappa} n^{2\kappa}) > n^\kappa$ for sufficiently large values of n . This concludes the proof. \square

We have thus far proved that the 1F-CSP problem fools the Sherali-Adams relaxation even after n^κ many rounds for some constant $1 > \kappa > 0$.

4 LP-hardness of VERTEX COVER and INDEPENDENT SET

4.1 Reduction of LP relaxations

We will now briefly introduce a formal framework for reducing between problems that is a stripped down version of the framework due to Braun *et al*, with a few notational changes. The interested reader can read the details of the full original framework in [10].

We start with the definition of an optimization problem.

Definition 4.1. An *optimization problem* $\Pi = (\mathcal{S}, \mathfrak{I})$ consists of a (finite) set \mathcal{S} of feasible solutions and a set \mathfrak{I} of instances. Each instance $\mathcal{I} \in \mathfrak{I}$ specifies an objective function from \mathcal{S} to \mathbb{R}_+ . We will denote this objective function by $\text{Val}_{\mathcal{I}}$ for maximization problems, and $\text{Cost}_{\mathcal{I}}$ for minimization problems. We let $\text{OPT}(\mathcal{I}) := \max_{S \in \mathcal{S}} \text{Val}_{\mathcal{I}}(S)$ for a maximization problem and $\text{OPT}(\mathcal{I}) := \min_{S \in \mathcal{S}} \text{Cost}_{\mathcal{I}}(S)$ for a minimization problem.

With this in mind we can give a general definition of the notion of an LP relaxation of an optimization problem Π . We deal with minimization problems first.

Definition 4.2. Let $\rho \geq 1$. A *factor- ρ LP relaxation* (or *ρ -approximate LP relaxation*) for a minimization problem $\Pi = (\mathcal{S}, \mathfrak{I})$ is a linear system $Ax \geq b$ with $x \in \mathbb{R}^d$ together with the following realizations:

- (i) **Feasible solutions** as vectors $x^S \in \mathbb{R}^d$ for every $S \in \mathcal{S}$ so that

$$Ax^S \geq b \quad \text{for all } S \in \mathcal{S}$$

(ii) **Objective functions** via affine functions $f_I : \mathbb{R}^d \rightarrow \mathbb{R}$ for every $I \in \mathfrak{I}$ such that

$$f_I(x^S) = \text{Cost}_I(S) \quad \text{for all } S \in \mathcal{S}$$

(iii) **Achieving approximation guarantee ρ** via requiring

$$\text{OPT}(I) \leq \rho \text{LP}(I) \quad \text{for all } I \in \mathfrak{I}$$

where $\text{LP}(I) := \min \{f_I(x) \mid Ax \geq b\}$.

Similarly, one can define factor- ρ LP relaxations of a maximization problem for $\rho \geq 1$. In our context, the concept of a (c, s) -approximate LP relaxation will turn out to be most useful. Here, c is the *completeness* and $s \leq c$ is the *soundness*. For a maximization problem, this corresponds to replacing condition (iii) above with

(iii)' **Achieving approximation guarantee (c, s)** via requiring

$$\text{OPT}(I) \leq s \implies \text{LP}(I) \leq c \quad \text{for all } I \in \mathfrak{I}.$$

The *size* of an LP relaxation is the number of inequalities in $Ax \geq b$. We let $\mathbf{fc}_+(\Pi, \rho)$ denote the minimum size of a factor- ρ LP relaxation for Π . In the terminology of [10], this is the ρ -approximate LP formulation complexity of Π . We define $\mathbf{fc}_+(\Pi, c, s)$ similarly.

In this framework problems can be naturally reduced to each other. We will use the following restricted form of reductions.

Definition 4.3. Let $\Pi_1 = (\mathcal{S}_1, \mathfrak{I}_1)$ be a maximization problem and $\Pi_2 = (\mathcal{S}_2, \mathfrak{I}_2)$ be a minimization problem. A *reduction from Π_1 to Π_2* consists of two maps, one $\mathcal{I}_1 \mapsto \mathcal{I}_2$ from \mathfrak{I}_1 to \mathfrak{I}_2 and the other $\mathcal{S}_1 \mapsto \mathcal{S}_2$ from \mathcal{S}_1 to \mathcal{S}_2 , subject to

$$\text{Val}_{\mathcal{I}_1}(S_1) = \mu_{\mathcal{I}_1} - \zeta_{\mathcal{I}_1} \cdot \text{Cost}_{\mathcal{I}_2}(S_2) \quad \mathcal{I}_1 \in \mathfrak{I}_1, S_1 \in \mathcal{S}_1$$

where $\mu_{\mathcal{I}_1}$ is called the *affine shift* and $\zeta_{\mathcal{I}_1} \geq 0$ is a normalization factor.

We say that the reduction is *exact* if additionally

$$\text{OPT}(\mathcal{I}_1) = \mu_{\mathcal{I}_1} - \zeta_{\mathcal{I}_1} \cdot \text{OPT}(\mathcal{I}_2) \quad \mathcal{I}_1 \in \mathfrak{I}_1.$$

The following result is a special case of a more general result by [10]. We give a proof for completeness.

Theorem 4.4. Let Π_1 be a maximization problem and let Π_2 be a minimization problem. Suppose that there exists an exact reduction from Π_1 to Π_2 with $\mu := \mu_{\mathcal{I}_1}$ constant for all $\mathcal{I}_1 \in \mathfrak{I}_1$. Then, $\mathbf{fc}_+(\Pi_1, c_1, s_1) \leq \mathbf{fc}_+(\Pi_2, \rho_2)$ where $\rho_2 := \frac{\mu - s_1}{\mu - c_1}$ (assuming $\mu > c_1 \geq s_1$).

Proof. Let $Ax \geq b$ by a ρ_2 -approximate LP relaxation for $\Pi_2 = (\mathcal{S}_2, \mathfrak{I}_2)$, with realizations x^{S_2} for $S_2 \in \mathcal{S}_2$ and $f_{\mathcal{I}_2} : \mathbb{R}^d \rightarrow \mathbb{R}$ for $\mathcal{I}_2 \in \mathfrak{I}_2$. We use the same system $Ax \geq b$ to define a (c_1, s_1) -approximate LP relaxation of the same size for $\Pi_1 = (\mathcal{S}_1, \mathfrak{I}_1)$ by letting $x^{S_1} := x^{S_2}$ where S_2 is the solution of Π_2 corresponding to $S_1 \in \mathcal{S}_1$ via the reduction, and similarly $f_{\mathcal{I}_1} := \mu - \zeta_{\mathcal{I}_1} f_{\mathcal{I}_2}$ with $\zeta_{\mathcal{I}_1} \geq 0$ where \mathcal{I}_2 is the instance of Π_2 to which \mathcal{I}_1 is mapped by the reduction and μ is the affine shift independent of the instance \mathcal{I}_1 .

Then conditions (i) and (ii) of Definition 4.3 are automatically satisfied. It suffices to check (iii)' with our choice of ρ_2 , for the given completeness c_1 and soundness s_1 . Assume that $\text{OPT}(\mathcal{I}_1) \leq s_1$ for some instance \mathcal{I}_1 of Π_1 . Then

$$\begin{aligned}
\text{LP}(\mathcal{I}_1) &= \mu - \zeta_{\mathcal{I}_1} \text{LP}(\mathcal{I}_2) && \text{(by definition of } f_{\mathcal{I}_1}, \text{ and since } \zeta_{\mathcal{I}_1} \geq 0) \\
&\leq \mu - \frac{1}{\rho_2} \cdot \zeta_{\mathcal{I}_1} \cdot \text{OPT}(\mathcal{I}_2) && \text{(since } \text{OPT}(\mathcal{I}_2) \leq \rho_2 \text{LP}(\mathcal{I}_2)) \\
&= \mu + \frac{\mu - c_1}{\mu - s_1} \cdot \underbrace{(\text{OPT}(\mathcal{I}_1) - \mu)}_{\leq s_1} && \text{(since the reduction is exact)} \\
&\leq \mu + \frac{\mu - c_1}{\mu - s_1} \cdot (s_1 - \mu) \\
&= c_1,
\end{aligned}$$

as required. Thus $Ax \geq b$ gives a (c_1, s_1) -approximate LP relaxation of Π_1 . The theorem follows. \square

We will also derive inapproximability of INDEPENDENT SET from a reduction between maximization problems. In this case the inapproximability factor obtained is of the form $\rho_2 = \frac{\mu + c_1}{\mu + s_1}$.

4.2 Hardness for VERTEX COVER and INDEPENDENT SET

We will now reduce 1F-CSP to VERTEX COVER with the reduction mechanism outlined in the previous section, which will yield the desired LP hardness for the latter problem.

We start by recasting VERTEX COVER, INDEPENDENT SET and 1F-CSP in our language. The two first problems are defined on a fixed graph $G = (V, E)$.

Problem 4.5 (VERTEX COVER(G)). The set of feasible solutions \mathcal{S} consists of all possible vertex covers $U \subseteq V$, and there is one instance $\mathcal{I} = \mathcal{I}(H) \in \mathfrak{I}$ for each induced subgraph H of G . For each vertex cover U we have $\text{Cost}_{\mathcal{I}(H)}(U) := |U \cap V(H)|$ being the size of the induced vertex cover in H .

Note that the instances we consider have 0/1 costs, which makes our final result stronger: even restricting to 0/1 costs does not make it easier for LPs to approximate VERTEX COVER. Similarly, for the independent set problem we have:

Problem 4.6 (INDEPENDENT SET(G)). The set of feasible solutions \mathcal{S} consists of all possible independent sets of G , and there is one instance $\mathcal{I} = \mathcal{I}(H) \in \mathfrak{I}$ for each induced subgraph H of G . For each independent set $I \in \mathcal{S}$, we have that $\text{Val}_{\mathcal{I}(H)}(I) := |I \cap V(H)|$ is the size of the induced independent set of H .

Finally, we can recast 1F-CSP as follows. Let $n, k \in \mathbb{N}$ be fixed, with $k \leq n$.

Problem 4.7 (1F-CSP(n, k)). The set of feasible solutions \mathcal{S} consists of all possible variable assignments, i.e., the vertices of the n -dimensional 0/1 hypercube and there is one instance $\mathcal{I} = \mathcal{I}(\mathcal{P})$ for each possible set $\mathcal{P} = \{P_1, \dots, P_m\}$ of one free bit predicates of arity k . As before, for an instance $\mathcal{I} \in \mathfrak{I}$ and an assignment $x \in \{0, 1\}^n$, $\text{Val}_{\mathcal{I}}(x)$ is the fraction of predicates P_i that x satisfies (see Definition 2.1).

With the notion of LP relaxations and 1F-CSP from above we can now formulate LP-hardness of approximation for 1F-CSPs, which follows directly from Corollary 3.7 by the result of [13].

Theorem 4.8. For every $\varepsilon > 0$ there exists a constant arity $k = k(\varepsilon)$ such that for infinitely many n we have $\mathbf{fc}_+(1\text{F-CSP}(n, k), 1 - \varepsilon, \varepsilon) \geq n^{\Omega(\log n / \log \log n)}$.

Following the approach in [10], we define a graph G over which we consider VERTEX COVER, which will correspond to our (family of) hard instances. This graph is a *universal* FGLSS graph as it encodes all possible choices of predicates simultaneously [19]. The constructed graph is similar to the one in [10], however now we consider *all* one free bit predicates and not just the MAX CUT predicate $x \oplus y$.

Definition 4.9 (VERTEX COVER host graph). For fixed number of variables n and arity $k \leq n$ we define a graph $G^* = G^*(n, k)$ as follows. Let x_1, \dots, x_n denote the variables of the CSP.

Vertices: For every one free bit predicate P of arity k and subset of indices $S \subseteq [n]$ of size k we have two vertices $v_{P,S,1}$ and $v_{P,S,2}$ corresponding to the two satisfying partial assignments for P on variables x_i with $i \in S$. For simplicity we identify the partial assignments with the respective vertices in G^* . Thus a partial assignment $\alpha \in \{0, 1\}^S$ satisfying predicate P has a corresponding vertex $v_{P,\alpha} \in \{v_{P,S,1}, v_{P,S,2}\}$.

Edges: Two vertices v_{P,α_1} and v_{P,α_2} are connected if and only if the corresponding partial assignments α_1 and α_2 are incompatible, i.e., there exists $i \in S_1 \cap S_2$ with $\alpha_1(i) \neq \alpha_2(i)$.

Note that the graph has $2\binom{k}{2}\binom{n}{k}$ vertices, which is polynomial in n for fixed k . In order to establish LP-inapproximability of VERTEX COVER and INDEPENDENT SET it now suffices to define a reduction satisfying Theorem 4.4.

Main Theorem 4.10. For every $\varepsilon > 0$ and for infinitely many n , there exists a graph G with $|V(G)| = n$ such that $\mathbf{fc}_+(\text{VERTEX COVER}(G), 2 - \varepsilon) \geq n^{\Omega(\log n / \log \log n)}$, and also $\mathbf{fc}_+(\text{INDEPENDENT SET}(G), 1/\varepsilon) \geq n^{\Omega(\log n / \log \log n)}$.

Proof. We reduce 1F-CSP on n variables with sufficiently large arity $k = k(\varepsilon)$ to VERTEX COVER over $G := G^*(n, k)$. For a 1F-CSP instance $\mathcal{I}_1 := \mathcal{I}_1(\mathcal{P})$ and set of predicates $\mathcal{P} = \{P_1, \dots, P_m\}$, let $H(\mathcal{P})$ be the induced subgraph of G on the set of vertices $V(\mathcal{P})$ corresponding to the partial assignments satisfying some constraint in \mathcal{P} . So $V(\mathcal{P}) = \{v_{P,S,i} \mid P \in \mathcal{P}, S \subseteq [n], |S| \leq k, i = 1, 2\}$.

In Theorem 4.8 we have shown that no LP of size at most $n^{o(\log n / \log \log n)}$ can provide an $(1 - \varepsilon, \varepsilon)$ -approximation for 1F-CSP for any $\varepsilon > 0$, provided the arity k is large enough. To prove that every LP relaxation with $2 - \varepsilon$ approximation guarantee for VERTEX COVER has size at least $n^{\Omega(\log n / \log \log n)}$, we provide maps defining a reduction from 1F-CSP to VERTEX COVER.

In the following, let $\Pi_1 = (\mathcal{S}_1, \mathfrak{S}_1)$ be the 1F-CSP problem and let $\Pi_2 = (\mathcal{S}_2, \mathfrak{S}_2)$ be the VERTEX COVER problem. In view of Definition 4.3, we map $\mathcal{I}_1 = \mathcal{I}_1(\mathcal{P})$ to $\mathcal{I}_2 = \mathcal{I}_2(H(\mathcal{P}))$ and let $\mu := 2$ and $\zeta_{\mathcal{I}_1} := \frac{1}{m}$ where m is the number of constraints in \mathcal{P} .

For a total assignment $x \in \mathcal{S}_1$ we define $U = U(x) := \{v_{P,\alpha} \mid \alpha \text{ satisfies } P \text{ and } x \text{ does not extend } \alpha\}$. The latter is indeed a vertex cover: we only have edges between conflicting partial assignments, and all the partial assignments that agree with x are compatible with each other. Thus $I = I(x) := \{v_{P,\alpha} \mid \alpha \text{ satisfies } P \text{ and } x \text{ extends } \alpha\}$ is an independent set and its complement U is a vertex cover.

We first verify the condition that $\text{Val}_{\mathcal{I}_1}(x) = 2 - \frac{1}{m} \text{Cost}_{\mathcal{I}_2}(U(x))$ for all instances $\mathcal{I}_1 \in \mathfrak{S}_1$ and assignments $x \in \mathcal{S}_1$. Every predicate P in \mathcal{P} over the variables in $\{x_i \mid i \in S\}$ has exactly two representative vertices $v_{P,\alpha_1}, v_{P,\alpha_2}$ where the $\alpha_1, \alpha_2 \in \{0, 1\}^S$ are the two partial assignments satisfying P . If an assignment $x \in \mathcal{S}_1$ satisfies the predicate P , then exactly one of α_1, α_2 is compatible with x . Otherwise, when $P(x) = 0$, neither of α_1, α_2 do. This means that in the former case exactly one of

$v_{P,\alpha_1}, v_{P,\alpha_2}$ is contained in U and in the latter both v_{P,α_1} and v_{P,α_2} are contained in U . It follows that for any $\mathcal{I}_1 = \mathcal{I}_1(\mathcal{P}) \in \mathfrak{I}_1$ and $x \in \mathcal{S}_1$ it holds

$$\text{Val}_{\mathcal{I}_1}(x) = 2 - \frac{1}{m} \text{Cost}_{\mathcal{I}_2}(U(x)).$$

In other words, for any specific \mathcal{P} the affine shift is 2, and the normalization factor is $\frac{1}{m}$.

Next we verify exactness of the reduction, i.e.,

$$\text{OPT}(\mathcal{I}_1) = 2 - \frac{1}{m} \text{OPT}(\mathcal{I}_2).$$

For this take an arbitrary vertex cover $U \in \mathcal{S}_2$ of G and consider its complement. This is an independent set, say I . As I is an independent set, all partial assignments α such that $v_{P,\alpha} \in I$ are compatible and there exists a total assignment x that is compatible with each α with $v_{P,\alpha} \in I$. Then the corresponding vertex cover $U(x)$ is contained in U . Thus there always exists an optimum solution to \mathcal{I}_2 that is of the form $U(x)$. Therefore, the reduction is exact.

It remains to compute the inapproximability factor via Theorem 4.4. We have

$$\rho_2 = \frac{2 - \varepsilon}{2 - (1 - \varepsilon)} \geq 2 - 3\varepsilon$$

A similar reduction works for INDEPENDENT SET. This time, the affine shift is $\mu = 0$ and we get an inapproximability factor of

$$\rho_2 = \frac{1 - \varepsilon}{\varepsilon} \geq \frac{1}{2\varepsilon}$$

for ε small enough. □

5 Upper bounds

Here we give a size- $O(n)$ LP relaxation for approximating INDEPENDENT SET within a factor- $O(\sqrt{n})$, which follows directly by work of Feige and Jozeph [21]. Note that this is strictly better than the $n^{1-\varepsilon}$ hardness obtained assuming $P \neq NP$ by [26]. This is possible because the *construction* of our LP is NP-hard while being still of small size, which is allowed in our framework.

Start with a greedy coloring of $G = (V, E)$: let I_1 be any maximum size independent set of G , let I_2 be any maximum independent set of $G - I_1$, and so on. In general, I_{j+1} is any maximum independent set of $G - I_1 - \dots - I_j$. Stop as soon as $I_1 \cup \dots \cup I_j$ covers the whole vertex set. Let $k \leq n$ denote the number of independent sets constructed, that is, the number of colors in the greedy coloring.

Feige and Jozeph [21] made the following observation:

Lemma 5.1. *Every independent set I of G has a nonempty intersection with at most $\lfloor 2\sqrt{n} \rfloor$ of the color classes I_j .*

Now consider the following linear constraints in $\mathbb{R}^V \times \mathbb{R}^k \simeq \mathbb{R}^{n+k}$:

$$0 \leq x_v \leq y_j \leq 1 \quad \forall j \in [k], v \in I_j \tag{5.1}$$

$$\sum_{j=1}^k y_j \leq \lfloor 2\sqrt{n} \rfloor. \tag{5.2}$$

These constraints describe the feasible set of our LP for INDEPENDENT SET on G . Each independent set I of G is realized by a 0/1-vector (x^I, y^I) defined by $x_v^I = 1$ iff I contains vertex v and $y_j^I = 1$ iff I has a nonempty intersection with color class I_j . For an induced subgraph H of G , we let $f_{I(H)}(x, y) := \sum_{v \in V(H)} x_v$. By Lemma 5.1, (x^I, y^I) satisfies (5.1)–(5.2). Moreover, we clearly have $f_{I(H)}(x^I, y^I) = |I \cap V(H)|$. Let $\text{LP}(I(H)) := \max\{f_{I(H)}(x, y) \mid (5.1), (5.2)\} = \max\{\sum_{v \in V(H)} x_v \mid (5.1), (5.2)\}$.

Lemma 5.2. *For every induced subgraph H of G , we have*

$$\text{LP}(I(H)) \leq \lfloor 2\sqrt{n} \rfloor \text{OPT}(I(H)).$$

Proof. When solving the LP, we may assume $x_v = y_j$ for all $j \in [k]$ and all $v \in I_j$. Thus the LP can be rewritten

$$\max \left\{ \sum_{j=1}^k |I_j \cap V(H)| \cdot y_j \mid 0 \leq y_j \leq 1 \ \forall j \in [k], \sum_{j=1}^k y_j \leq \lfloor 2\sqrt{n} \rfloor \right\}.$$

Because the feasible set is a 0/1-polytope, we see that the optimum value of this LP is attained by letting $y_j = 1$ for at most $\lfloor 2\sqrt{n} \rfloor$ of the color classes I_j and $y_j = 0$ for the others. Thus some color class I_j has weight at least $1/\lfloor 2\sqrt{n} \rfloor$ of the LP value. \square

By Lemma 5.2, constraints (5.1)–(5.2) provide a size- $O(n)$ factor- $O(\sqrt{n})$ LP relaxation of INDEPENDENT SET.

Theorem 5.3. *For every n -vertex graph G , $\text{fc}_+(\text{INDEPENDENT SET}(G), 2\sqrt{n}) \leq O(n)$.*

Although the LP relaxation (5.1)–(5.2) is NP-hard to construct, it is allowed by our framework because we do not bound the time needed to construct the LP. To our knowledge, this is the first example of a polynomial-size extended formulation outperforming polynomial-time algorithms.

We point out that a factor- $n^{1-\varepsilon}$ LP-inapproximability of INDEPENDENT SET holds in a different model, known as the *uniform model* [11, 9]. In that model, we seek an LP relaxation that approximates *all* INDEPENDENT SET instances with the same number of vertices n . This roughly corresponds to solving INDEPENDENT SET by approximating the correlation polytope in some way, which turns out to be strictly harder than approximating the stable set polytope, as shown by our result above.

6 LP Hardness for q -UNIFORM-VERTEX-COVER

In order to prove LP lower bounds for VERTEX COVER and INDEPENDENT SET in Sections 3 and 4, we first started by providing a reduction from the UNIQUE GAMES problem to the 1F-CSP problem, that implied that no *small size* linear program is a $(1 - \varepsilon, \varepsilon)$ -approximation for the 1F-CSP problem. We then gave a gap-preserving reduction from any LP approximating 1F-CSP to any LP approximating VERTEX COVER, and showed that no small size LP can provide a $(2 - \varepsilon)$ -approximation for the VERTEX COVER problem.

Our approach for the q -UNIFORM-VERTEX-COVER will be similar, however our starting point is a CONSTRAINT SATISFACTION PROBLEM different than 1F-CSP. This new CONSTRAINT SATISFACTION PROBLEM, that we refer to as NOT-EQUAL-CSP, is defined as follows:

Definition 6.1. A CSP of arity k over the domain⁷ \mathbb{Z}_q is referred to as NOT-EQUAL-CSP if each constraint $P : \mathbb{Z}_q^k \rightarrow \{0, 1\}$ is of the following form

$$P_A(x_1, x_2, \dots, x_k) = 1 \quad \text{if and only if} \quad \bigwedge_{i=1}^k (x_i \neq a_i)$$

⁷For convenience, we use the additive group $\mathbb{Z}_q = \{0, \dots, q-1\}$ instead of $[q]$ as the domain of our CSP.

for some $A = (a_1, a_2, \dots, a_k) \in \mathbb{Z}_q^k$. When $x \in \mathbb{Z}_q^n$, for some $n \geq k$, a predicate $P := P_{S,A}$ is additionally indexed by a set $S = \{i_1, i_2, \dots, i_k\} \subseteq [n]$, and $P_{S,A}(x) = P_A(x_{i_1}, x_{i_2}, \dots, x_{i_k})$.

We remark that the above definition should not be confused with the common Not-All-Equal predicate.

Similar to the approach of VERTEX COVER, we shall prove that there is no small linear programming relaxation for q -UNIFORM-VERTEX-COVER with a good approximation guarantee in two steps. In the first step, we prove that no small linear programming relaxation can *approximate well* the NOT-EQUAL-CSP problem. We then give a gap-preserving reduction from this problem to that of q -UNIFORM-VERTEX-COVER in the framework of [10].

6.1 Sherali-Adams Integrality Gap for NOT-EQUAL-CSP

This section will be dedicated to proving the following theorem.

Theorem 6.2. *For any $\varepsilon > 0$ and integer $q \geq 2$, there exist $\kappa > 0$ and an integer k so that for infinitely many n there exists a NOT-EQUAL-CSP instance \mathcal{I} of arity k over n variables satisfying*

- $\text{OPT}(\mathcal{I}) \leq \varepsilon$;
- *There is a solution to the n^κ -round Sherali-Adams relaxation of value $1 - 1/q - \varepsilon$.*

The above theorem states that the NOT-EQUAL-CSP problem can fool the Sherali-Adams relaxation even after n^κ many rounds. Before we proceed, we discuss functions of the form $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$. These functions will play a crucial role in the analysis.

6.1.1 Functions Over the Domain \mathbb{Z}_q

In order to construct Sherali-Adams integrality gaps for the NOT-EQUAL-CSP problem, we also reduce from the UNIQUE GAMES problem. The analysis of this reduction relies heavily on known properties regarding functions of the form $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$, where \mathbb{Z}_q is to be thought of as the domain of the new CSP, and R as the label set size of the UNIQUE GAMES instance. More precisely, we exploit the drastic difference in the behavior of functions depending on whether they have *influential* coordinates or not. To quantify these differences, we first need the following definitions.

Definition 6.3. For a function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$, and an index $i \in [R]$, the influence of the i -th coordinate is given by

$$\text{Inf}_i(f) = \mathbb{E}[\text{Var}[f(x)|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]]$$

where x_1, \dots, x_n are uniformly distributed.

An alternative definition for the influence requires defining the Fourier expansion of a function f of the form $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$. To do this, let $\phi_0 \equiv 1, \phi_1, \dots, \phi_{q-1} : \mathbb{Z}_q \mapsto \mathbb{R}$ be such that for all $i, j \in [q]$, we have

$$\mathbb{E}_{y \in \mathbb{Z}_q} [\phi_i(y)\phi_j(y)] = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

where the expectation is taken over the uniform distribution, and define the functions $\phi_\alpha : \mathbb{Z}_q^R \mapsto \mathbb{R}$ for every $\alpha \in \mathbb{Z}_q^R$ to be

$$\phi_\alpha(x) := \prod_{i=1}^R \phi_{\alpha_i}(x_i)$$

for any $x \in \mathbb{Z}_q^R$. We take these functions for defining our Fourier basis. Note that this coincides with the boolean case, where for $b \in \{0, 1\}$ we have $\phi_0(b) \equiv 1$, and $\phi_1(b) = (-1)^b$ (or the identity function in the $\{-1, 1\}$ domain). For a more elaborate discussion on the Fourier expansion in generalized domains, we refer the interested reader to Chapter 8 in [41].

Having fixed the functions $\phi_0, \phi_1, \dots, \phi_{q-1}$, every function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$ can be uniquely expressed as

$$f(x) = \sum_{\alpha \in \mathbb{Z}_q^R} \hat{f}_\alpha \phi_\alpha(x)$$

Equipped with this, we can relate the influence of a variable $i \in [R]$ with respect to a function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$, to the Fourier coefficients of f as follows:

$$\text{Inf}_i(f) = \sum_{\alpha: \alpha_i \neq 0} \hat{f}_\alpha^2$$

In our analysis we will however be interested in *degree- d* influences, denoted $\text{Inf}_i^d(d)$ and defined as

$$\text{Inf}_i^d(f) = \sum_{\alpha: \alpha_i \neq 0, |\alpha| \leq d} \hat{f}_\alpha^2$$

where $|\alpha|$ in this context is the support of α , i.e., the number of indices $j \in [R]$ such that $\alpha_j \neq 0$.

Observation 6.4 (see, e.g., Proposition 3.8 in [40]). For a function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$, the sum of all degree- d influences is at most d .

We will also need a generalization of the notion of sub-cubes defined in Figure 1 in order to state the "It Ain't Over Till It's Over" Theorem [40], a main ingredient of the analysis of the reduction. In fact we only state and use a special case of it, as it appears in [52].

Definition 6.5. Fix $\varepsilon > 0$. For $x \in \mathbb{Z}_q^R$, and $S_\varepsilon \subseteq [R]$ such that $|S_\varepsilon| = \varepsilon R$, the sub-cube C_{x, S_ε} is defined as follows:

$$C_{x, S_\varepsilon} := \{z \in \mathbb{Z}_q^R : z_j = x_j \ \forall j \notin S_\varepsilon\}$$

Theorem 6.6 (*Special case of the It Ain't Over Till It's Over Theorem*). For every $\varepsilon, \delta > 0$ and integer q , there exist $\vartheta > 0$ and integers t, d such that any collection of functions $f_1, \dots, f_t : \mathbb{Z}_q^R \mapsto \{0, 1\}$ that satisfies

$$\forall j : \mathbb{E}[f_j] \geq \delta \quad \text{and} \quad \forall i \in [R], \forall 1 \leq \ell_1 \neq \ell_2 \leq t : \min\{\text{Inf}_i^d(f_{\ell_1}), \text{Inf}_i^d(f_{\ell_2})\} \leq \vartheta,$$

has the property

$$\mathbb{P}_{x, S_\varepsilon} \left[\bigwedge_{j=1}^t (f_j(C_{x, S_\varepsilon}) \equiv 0) \right] \leq \delta.$$

Essentially what this theorem says is that if a collection of t fairly balanced functions are all identical to zero on the same random sub-cube with non-negligible probability, then at least two of these functions must share a common influential coordinate. In fact all the functions that we use throughout this section satisfy a strong balance property, that we denote by *folding*.⁸

Folded Functions. We say that a function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$ is *folded* if every *line* of the form $\{x \in \mathbb{Z}_q^R \mid x = a + \lambda \mathbf{1}, \lambda \in \mathbb{Z}_q\}$ contains a unique point where $f(x)$ is zero, where $\mathbf{1} \in \mathbb{Z}_q^R$ is the all-one vector and $a \in \mathbb{Z}_q^R$ is any point.

Remark 6.7. For any folded function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$, we have that $\mathbb{E}_x[f(x)] = 1 - 1/q$.

We shall also extend the notion of *dictatorship* functions restricted to the folded setting. In this setting, the ℓ -th coordinate dictator function $f_\ell : \mathbb{Z}_q^R \mapsto \{0, 1\}$ for some $\ell \in [R]$ is defined as

$$f_\ell(x) = \begin{cases} 1 & \text{if } x_\ell \neq 0 \\ 0 & \text{if } x_\ell = 0. \end{cases}$$

Notice that f_ℓ is folded because it is zero exactly on the coordinate hyperplane $\{x \in \mathbb{Z}_q^R \mid x_\ell = 0\}$.

Truth Table Model. In order to guarantee the folding property of a function $f : \mathbb{Z}_q^R \mapsto \{0, 1\}$ in the truth table model, we adopt the following convention:

- The truth table Υ_f has q^{R-1} entries in \mathbb{Z}_q , one for each $x \in \mathbb{Z}_q^R$ such that $x_1 = 0$.
- For each $x \in \mathbb{Z}_q^R$ with $x_1 = 0$, the corresponding entry $\Upsilon_f(x)$ contains the unique $\lambda \in \mathbb{Z}_q$ such that $f(x + \lambda \mathbf{1}) = 0$.

We can however use Υ_f to query $f(x)$ for any $x \in \mathbb{Z}_q^R$ as follows: we have $f(x) = 0$ whenever $\Upsilon_f(x - x_1 \mathbf{1}) = \Upsilon_f(0, x_2 - x_1, \dots, x_R - x_1) = x_1$ and $f(x) = 1$ otherwise.

We can now readily extend the notion of the *long code* encoding to match our definition of dictatorship functions.

Definition 6.8. The long code encoding of an index $\ell \in [R]$ is simply Υ_{f_ℓ} , the truth table of the *folded* dictatorship function of the ℓ -th coordinate. Similarly, the long code $\Upsilon_{f_\ell} \in \mathbb{Z}_q^{q^{R-1}}$ is indexed by all $x \in \mathbb{Z}_q^R$ such that $x_1 = 0$.

6.1.2 Reduction from UNIQUE GAMES to NOT-EQUAL-CSP

We first describe the reduction from UNIQUE GAMES to NOT-EQUAL-CSP that is similar in many aspects to the reduction in Section 3.2. We then show that it also preserves the Sherali-Adams integrality gap.

Reduction. Let $\mathcal{U} = (G, [R], \Pi)$ be a UNIQUE GAMES instance over a regular bipartite graph $G = (V, W, E)$. Given \mathcal{U} , we construct an instance \mathcal{I} of NOT-EQUAL-CSP. The reduction has three parameters: an integer $q \geq 2$ and reals $\delta, \varepsilon > 0$, where ε is chosen such that εR is an integer. We then select t to be a large integer depending on ε, δ and q so as to satisfy Lemma 6.9.

⁸We abuse the notion of folding here, and we stress that this should not be confused with the usual notion of folding in the literature, although it coincides with standard folding for the boolean case.

The resulting NOT-EQUAL-CSP instance \mathcal{I} will be defined over $|W|q^{R-1}$ variables and $c|V|$ constraints, where $c := c(R, \varepsilon, t, \Delta, q)$ is a function of the degree Δ of the UNIQUE GAMES instance, and the constants R, t, q and ε . For our purposes, the UNIQUE GAMES integrality gap instance that we start from, has constant degree Δ , and hence c is a constant.

We refer to the variables of \mathcal{I} as follows: it has a variable $\langle w, z \rangle \in \mathbb{Z}_q$ for each $w \in W$ and $z \in \mathbb{Z}_q^R$ such that $z_1 = 0$. For further reference, we let $\text{Var}(\mathcal{I})$ denote the set of variables of \mathcal{I} . The constraints of \mathcal{I} are picked according the distribution in Figure 2 on page 21. One can see that a constraint $C := C(v, \mathcal{W}_v, x, S_\varepsilon)$ is then defined by the random vertex v (Line 1), the t random neighbors $\mathcal{W}_v = \{w_1, \dots, w_t\}$ (Line 2), the random $x \in \mathbb{Z}_q^R$ (Line 3) and the random subset $S_\varepsilon \subseteq [R]$ (Line 4).

1. Pick a vertex $v \in V$ uniformly at random.
2. Pick t vertices w_1, \dots, w_t randomly and independently from the neighborhood $N(v) = \{w \in W : vw \in E\}$.
3. Pick $x \in \mathbb{Z}_q^R$ at random.
4. Let $m = \varepsilon R$. Pick indices i_1, \dots, i_m randomly and independently from $[R]$ and let $S_\varepsilon = \{i_1, \dots, i_m\}$ be the set of those indices.
5. Output the constraint on the variables $\{\langle w_i, z - z_1 \mathbf{1} \rangle \mid i \in [t], \pi_{v, w_i}^{-1}(z) \in C_{x, S_\varepsilon}\}$ that is true if

$$\langle w_i, z - z_1 \mathbf{1} \rangle \neq z_1 \quad \forall 1 \leq i \leq t, \forall z \text{ such that } \pi_{v, w_1}^{-1}(z) \in C_{x, S_\varepsilon}$$

where $\pi(z)$ for $z \in \mathbb{Z}_q^R$ is defined as $\pi(z) := (z_{\pi(1)}, z_{\pi(2)}, \dots, z_{\pi(R)})$.

Figure 2: Distribution for the NOT-EQUAL-CSP constraints

Note that if we think of the variables $\langle w, z \rangle$ for a fixed $w \in W$ as the truth table of some function $f_w : \mathbb{Z}_q^R \mapsto \{0, 1\}$, then f is forced to satisfy the folding property.

We claim that if the starting UNIQUE GAMES instance \mathcal{U} was a Sherali-Adams integrality gap instance, then \mathcal{I} is also an integrality gap instance for the NOT-EQUAL-CSP problem. Similar to Section 3.2, we prove this in two steps; we first show that if $\text{OPT}(\mathcal{U})$ is *small*, then so is $\text{OPT}(\mathcal{I})$. Formally speaking, the following holds:

Lemma 6.9. *For every $\varepsilon, \eta > 0$ and alphabet size $q \geq 2$ there exists an integer t so that $\text{OPT}(\mathcal{I}) \leq \eta$ if $\text{OPT}(\mathcal{U}) \leq \delta$ where $\delta > 0$ is a constant that only depends on ε, η, q and t .*

Proof. Suppose towards contradiction that $\text{OPT}(\mathcal{I}) > \eta$. As noted earlier, for a fixed $w \in W$, we can think of the variables $\langle w, z \rangle \in \text{Var}(\mathcal{I})$ as the truth table of a folded function $f_w : \mathbb{Z}_q^R \mapsto \{0, 1\}$, where $\Upsilon_{f_w}(z) := \langle w, z \rangle$. This is possible since the variables $\langle w, z \rangle \in \text{Var}(\mathcal{I})$ are restricted to $z \in \mathbb{Z}_q^R$ with $z_0 = 0$. Given this alternative point of view, define for every vertex $w \in W$, a set of *candidate labels* $L[w]$ as follows:

$$L[w] = \{i \in [R] : \text{Inf}_i^d(f_w) \geq \vartheta\}$$

Note that $|L[w]| \leq d/\vartheta$ by Observation 6.4.

For every vertex $v \in V$, and every $\mathcal{W}_v = \{w_1, \dots, w_t\} \subseteq N(v)$, let

$$C_{v, \mathcal{W}_v} := \left\{ C_{v, \mathcal{W}_v, x, S} : x \in \mathbb{Z}_q^R, S \subseteq [R] \text{ such that } |S| = \varepsilon R \right\}$$

A standard counting argument then shows that if $\text{OPT}(\mathcal{I}) > \eta$, then at least $\eta/2$ fraction of the tuples (v, w_1, \dots, w_t) have at least $\eta/2$ fraction of the constraints inside C_{v, \mathcal{W}_v} satisfied. We refer to such tuples as *good*. Adopting the language of folded functions instead of variables, the aforementioned statement can be casted as

$$\mathbb{P}_{x \in \mathbb{Z}_q^R, S_\varepsilon \subseteq [R]} \left[\bigwedge_{i=1}^t (f_{w_i}(\pi_{v, w_i}(C_{x, S_\varepsilon})) \equiv 1) \right] \geq \eta/2 \quad \text{if the tuple } (v, w_1, \dots, w_t) \text{ is good}$$

where

$$f(\pi(C_{x, S_\varepsilon})) \equiv 1 \quad \iff \quad f(z) = 1 \quad \forall z \text{ such that } \pi^{-1}(z) \in C_{x, S_\varepsilon}$$

From Remark 6.7, we get that $\mathbb{E}[f_{w_i}] = 1 - 1/q$, and hence invoking Theorem 6.6 on the functions $\tilde{f}_{w_1}, \dots, \tilde{f}_{w_t}$, where $\tilde{f}_{w_i}(x) := 1 - f_{w_i}(\pi_{v, w_i}(x))$, yields that for every good tuple, there exists $\ell_1 \neq \ell_2 \in \{1, 2, \dots, t\}$ such that $\tilde{f}_{w_{\ell_1}}$ and $\tilde{f}_{w_{\ell_2}}$ share a common influential coordinate. Note that this is equivalent to saying that there exists $j_1 \in L[w_{\ell_1}]$, $j_2 \in L[w_{\ell_2}]$ such that $\pi_{v, w_{\ell_1}}(j_1) = \pi_{v, w_{\ell_2}}(j_2)$.

We now claim that if $\text{OPT}(\mathcal{I}) > \eta$, then we can come up with a labeling $\Lambda : V \cup W \mapsto [R]$ that satisfies at least $\frac{\eta \vartheta^2}{2d^2 t^2}$ of edges, which contradicts the fact that $\text{OPT}(\mathcal{U}) \leq \delta$ for a small enough value of $\delta > 0$. Towards this end, consider the following randomized labeling procedure:

1. For every $w \in W$, let $\Lambda(w)$ be a random label from the set $L[w]$, or an arbitrary label if $L[w] = \emptyset$.
2. For every $v \in V$, pick a random neighbor $w \in N(v)$ and set $\Lambda(v) = \pi_{v, w}(\Lambda(w))$.

We can readily calculate the fraction of edges in \mathcal{U} that are satisfied by Λ . This follows from putting the following observations together:

1. If we pick a random tuple (v, w_1, \dots, w_t) , it is *good* with probability $\eta/2$.
2. If (v, w_1, \dots, w_t) is *good*, and we pick w', w'' at random from $\{w_1, \dots, w_t\}$, then with probability $1/t^2$ the functions $f_{w'}$ and $f_{w''}$ share a common influential coordinates.
3. If (v, w_1, \dots, w_t) is *good*, and the functions $f_{w'}$ and $f_{w''}$ share a common influential coordinates, then picking a random label to w' and w'' from $L[w']$ and $L[w'']$ respectively, will satisfies $\pi_{v, w'}(\Lambda(w')) = \pi_{v, w''}(\Lambda(w''))$ with probability $1/(d^2/\vartheta^2)$.

Hence the expected number of edges satisfied by Λ in this case is

$$\mathbb{P}_{v, w \in E} [\Lambda(v) = \pi_{v, w}(\Lambda(w))] = \frac{\eta \vartheta^2}{2d^2 t^2}$$

□

We now show that given an r -rounds Sherali-Adams solution of *high* value for \mathcal{U} , we can also come up with an r -rounds Sherali-Adams solution for \mathcal{I} of high value as well. The proof goes along the same lines of that of Lemma 3.5, and hence we will try to only highlight the differences.

Lemma 6.10. Let $\{\mu(S) \mid S \subseteq V \cup W, |S| \leq r\}$ be a consistent collection of local distributions defining a solution to the r -rounds Sherali-Adams relaxation of the regular bipartite **UNIQUE GAMES** instance \mathcal{U} . Then we can define a consistent collection of local distributions $\{\sigma(S) \mid S \subseteq \text{Var}(\mathcal{I}), |S| \leq r\}$ defining a solution to the r -rounds Sherali-Adams relaxation of the **NOT-EQUAL-CSP** instance \mathcal{I} so that

$$\mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \sigma(S_C)} [\alpha \text{ satisfies } C] \right] \geq (1 - \varepsilon)(1 - \frac{1}{q}) \left(1 - t \cdot \mathbb{E}_{vw \in E} \left[\mathbb{P}_{(\Lambda(v), \Lambda(w)) \sim \mu(\{v, w\})} [\Lambda(v) \neq \pi_{w,v}(\Lambda(w))] \right] \right),$$

where t and ε are the parameters of the reduction, and $\sigma(S_C)$ is the distribution over the set of variables in the support S_C of constraint C .

Proof. Let $\{\mu(S) \mid S \subseteq V \cup W, |S| \leq r\}$ be a solution to the r -rounds SA relaxation of the **UNIQUE GAMES** instance \mathcal{U} , and recall that \mathcal{I} is the **NOT-EQUAL-CSP** instance we get by applying the reduction. We will now use the collection of consistent local distributions of the **UNIQUE GAMES** instance, to construct another collection of consistent local distributions for the variables in $\text{Var}(\mathcal{I})$.

For every set $S \subseteq \text{Var}(\mathcal{I})$ such that $|S| \leq r$, let $T_S \subseteq W$ be the subset of vertices in the **UNIQUE GAMES** instance defined as follows:

$$T_S = \{w \in W : \langle w, x \rangle \in S\}. \quad (6.1)$$

We will now construct $\sigma(S)$ from $\mu(T_S)$ in the following manner. Given a labeling Λ_{T_S} for the vertices in T_S drawn from $\mu(T_S)$, define an assignment α_S for the variables in S as follows: for a variable $\langle w, x \rangle \in S$, let $\ell = \Lambda_{T_S}(w)$ be the label of w according to Λ_{T_S} . Then the new assignment α_S sets $\alpha_S(\langle w, x \rangle) := \Upsilon_{f_\ell}(x)$, where Υ_{f_ℓ} is the long code encoding of ℓ as in Definition 6.8. The aforementioned procedure defines a family $\{\sigma(S)\}_{S \subseteq \text{Var}(\mathcal{I}), |S| \leq r}$ of local distributions for the variables of the **NOT-EQUAL-CSP** instance \mathcal{I} . The same argument as in the proof of Lemma 3.5 yields that $\{\sigma(S) \mid S \subseteq \text{Var}(\mathcal{I}), |S| \leq r\}$ defines a feasible solution for the r -round Sherali-Adams relaxation of the **NOT-EQUAL-CSP** instance \mathcal{I} .

It remains to bound the value of this feasible solution, i.e.,

$$\mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \sigma(S_C)} [\alpha \text{ satisfies } C] \right] = \mathbb{E}_{v, w_1, \dots, w_t} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\}), x, S} [\psi(\Lambda) \text{ satisfies } C(v, \mathcal{W}_v, x, S)] \right]. \quad (6.2)$$

where $\psi(\cdot)$ the operator mapping a labeling of the vertices in T_S to an assignment for the variables in S , i.e., $\psi(\Lambda_{T_S}) = \alpha_S$. The following claim, which is in some sense the equivalent of Claim 3.6 in the **NOT-EQUAL-CSP** language, along with the same remaining steps of the proof of Lemma 3.5 will yield the proof.

Claim 6.11. If Λ satisfies vw_1, \dots, vw_t simultaneously, then $\psi(\Lambda)$ satisfies $C(v, \mathcal{W}_v, x, S)$ with probability at least $(1 - \varepsilon)(1 - \frac{1}{q})$. Moreover, if we *additionally* have that $\Lambda(v) \notin S$ and $x_{\Lambda(v)} \neq 0$, then $\psi(\Lambda)$ always satisfies $C(v, \mathcal{W}_v, x, S)$.

Equipped with this, we can use conditioning to lower-bound the probability inside the expectation in (6.2) by a product of two probabilities, where the first is

$$\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\}), x, S} [\psi(\Lambda) \text{ satisfies } C(v, \mathcal{W}_v, x, S) \mid \Lambda \text{ satisfies } vw_1, \dots, vw_t] \quad (6.3)$$

and the second is

$$\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\})} [\Lambda \text{ satisfies } vw_1, \dots, vw_t].$$

Thus using Claim 6.11, we get

$$\begin{aligned} \mathbb{E}_{C \in \mathcal{C}} \left[\mathbb{P}_{\alpha \sim \sigma(S_C)} [\alpha \text{ satisfies } C] \right] &\geq (1 - \varepsilon) \left(1 - \frac{1}{q}\right) \cdot \mathbb{E}_{v, w_1, \dots, w_t} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w_1, \dots, w_t\})} [\Lambda \text{ satisfies } vw_1, \dots, vw_t] \right] \\ &\geq (1 - \varepsilon) \left(1 - \frac{1}{q}\right) \cdot \left(1 - t \cdot \mathbb{E}_{v, w} \left[\mathbb{P}_{\Lambda \sim \mu(\{v, w\})} [\Lambda \text{ does not satisfy } vw] \right] \right) \end{aligned}$$

□

The proof of Corollary 3.7 adjusted to the NOT-EQUAL-CSP problem now yields Theorem 6.2.

6.2 LP-reduction from NOT-EQUAL-CSP to q -UNIFORM-VERTEX-COVER

We will now reduce NOT-EQUAL-CSP to q -UNIFORM-VERTEX-COVER on q -Uniform hypergraphs with the reduction mechanism outlined in Section 4.1, which will yield the desired LP hardness for the latter problem.

We start by recasting q -UNIFORM-VERTEX-COVER and NOT-EQUAL-CSP in the language of Section 4.1. The first problem is defined on a fixed q -uniform hypergraph $H = (V, E)$.

Problem 6.12 (q -UNIFORM-VERTEX-COVER(G)). The set of feasible solutions \mathcal{S} consists of all possible vertex covers $U \subseteq V$, and there is one instance $\mathcal{I} = \mathcal{I}(H') \in \mathfrak{I}$ for each induced subgraph H' of G . For each vertex cover U we have $\text{Cost}_{\mathcal{I}(H')}(U) := |U \cap V(H')|$ being the size of the induced vertex cover in H' .

We also recast NOT-EQUAL-CSP as follows. Let $n, q, k \in \mathbb{N}$ be fixed, with $k \leq n$.

Problem 6.13 (NOT-EQUAL-CSP(n, q, k)). The set of feasible solutions \mathcal{S} consists of all possible variable assignments, i.e., all possible values of \mathbb{Z}_q^n and there is one instance $\mathcal{I} = \mathcal{I}(\mathcal{P})$ for each possible set $\mathcal{P} = \{P_1, \dots, P_m\}$ of NOT-EQUAL-CSP predicates of arity k . As before, for an instance $\mathcal{I} \in \mathfrak{I}$ and an assignment $x \in \mathbb{Z}_q^n$, $\text{Val}_{\mathcal{I}}(x)$ is the fraction of predicates P_i that x satisfies (see Definition 6.1).

With the notion of LP relaxations and NOT-EQUAL-CSP from above, we can now formulate LP-hardness of approximation for NOT-EQUAL-CSPs, which follows directly from Theorem 6.2 by the result of [13] (See the discussion in [13] and Section 7 in [37]).

Theorem 6.14. *For every $\varepsilon > 0$ and alphabet size $q \geq 2$, there exists a constant arity $k = k(\varepsilon)$ such that for infinitely many n we have $\mathbf{fc}_+(\text{NOT-EQUAL-CSP}(n, q, k), 1 - 1/q - \varepsilon, \varepsilon) \geq n^{\Omega(\log n / \log \log n)}$.*

Similar to Section 4.2, we first define our host hypergraph, and then provide a reduction that will yield our hardness result for q -UNIFORM-VERTEX-COVER using Theorem 4.4.

Definition 6.15 (q -UNIFORM-VERTEX-COVER host hypergraph). For fixed number of variables n , alphabet q , and arity $k \leq n$ we define a hypergraph $H^* = H^*(n, q, k)$ as follows. Let x_1, \dots, x_n denote the variables of the CSP.

Vertices: For every subset $S = \{i_1, \dots, i_k\} \subseteq [n]$, and every value of $A = (a_1, \dots, a_k) \in \mathbb{Z}_q^k$, we have a vertex $v_{S,A}$ corresponding to the NOT-EQUAL-CSP predicate

$$P(x_{i_1}, \dots, x_{i_k}) = 1 \quad \text{if and only if} \quad \bigwedge_{j=1}^k (x_{i_j} \neq a_j)$$

Hyperedges: Any q vertices $v_{S_1, A_1}, \dots, v_{S_q, A_q}$ are connected with a hyperedge if there exists a variable $x_i \in \bigcap_{j=1}^q S_j$, such that $a_{i_1} \neq a_{i_2} \neq \dots \neq a_{i_q}$, where a_{i_j} is the entry of the vector A_j that is compared versus the variable x_i in the predicate defined by the pair (S_j, A_j) . In other words, we have a hyperedge connecting q vertices sharing a *common variable* x_i , if no two of their *corresponding* predicates check x_i versus the same $a \in \mathbb{Z}_q$.

Note that the graph has $q^k \binom{n}{k}$ vertices, which is polynomial in n for fixed k and q . In order to establish LP-inapproximability of q -UNIFORM-VERTEX-COVER it now suffices to define a reduction satisfying Theorem 4.4.

Main Theorem 6.16. *For every $\varepsilon > 0, q \geq 2$ and for infinitely many n , there exists a hypergraph H with $|V(H)| = n$ such that $\mathbf{fc}_+(q\text{-UNIFORM-VERTEX-COVER}(H), q - \varepsilon) \geq n^{\Omega(\log n / \log \log n)}$.*

Proof. We reduce NOT-EQUAL-CSP on n variables of alphabet \mathbb{Z}_q with sufficiently large arity $k = k(\varepsilon)$ to q -UNIFORM-VERTEX-COVER over $H := H^*(n, q, k)$. For a NOT-EQUAL-CSP instance $I_1 = I_1(\mathcal{P})$ and set of Not-Equal predicates $\mathcal{P} = \{P_{S_1, A_1}, P_{S_2, A_2}, \dots, P_{S_m, A_m}\}$, let $H(\mathcal{P})$ be the induced subgraph of G on the set of vertices $V(\mathcal{P}) = \{v_{S_i, A_i} \mid 1 \leq i \leq m\}$.

Similarly to Section 4.2, we provide maps defining a reduction from NOT-EQUAL-CSP to q -UNIFORM-VERTEX-COVER. The proof will then follow by combining Theorems 6.14 and 4.4.

In the following, let $\Pi_1 = (S_1, \mathfrak{S}_1)$ be the NOT-EQUAL-CSP problem and let $\Pi_2 = (S_2, \mathfrak{S}_2)$ be the q -UNIFORM-VERTEX-COVER problem. In view of Definition 4.3, we map $I_1 = I_1(\mathcal{P})$ to $I_2 = I_2(H(\mathcal{P}))$ and let $\mu := 1$ and $\zeta_{I_1} := \frac{1}{m}$ where m is the number of constraints in \mathcal{P} .

For a total assignment $x \in S_1$ we define $U = U(x) := \{v_{S, A} : P_{S, A}(x) = 0\}$. The latter is indeed a vertex cover. To see this, consider its complement $I = I(x) := \{v_{S, A} \mid P_{S, A}(x) = 1\}$. Since x satisfies all the constraints corresponding to vertices in I simultaneously, no hyperedge can be completely contained in I . Otherwise this would imply that there exists a variable x_i , and q predicates $P'_1, P'_2, \dots, P'_q \in \mathcal{P}$ requiring $x_i \neq j$ for all $j \in \mathbb{Z}_q$, and yet are all simultaneously satisfied by x .

We first verify the condition that $\text{Val}_{I_1}(x) = 1 - \frac{1}{m} \text{Cost}_{I_2}(U(x))$ for all instances $I_1 \in \mathfrak{S}_1$ and assignments $x \in S_1$. Every predicate $P_{S, A}$ in \mathcal{P} over the variables in $\{x_i \mid i \in S\}$ has exactly one representative vertex $v_{S, A}$, that will be inside U only if $P_{S, A}(x) = 0$, and hence our claim holds. In other words, for any specific \mathcal{P} the affine shift is 1, and the normalization factor is $\frac{1}{m}$.

Next we verify exactness of the reduction, i.e.,

$$\text{OPT}(I_1) = 1 - \frac{1}{m} \text{OPT}(I_2).$$

For this take an arbitrary vertex cover $U \in S_2$ of H and consider its complement. This is an independent set, say I . As I is an independent set⁹, we know that for any variable x_ℓ in $\bigcup_{v_{S, A} \in I} S$, there exist a least one $\tilde{a}_{x_\ell} \in \mathbb{Z}_q$ such that x_ℓ is *not checked* versus \tilde{a}_{x_ℓ} in any of the predicates corresponding to vertices in I . Hence any assignment x setting each x_ℓ to \tilde{a}_{x_ℓ} as defined earlier, sets $P_{S, A}(x) = 1$ for all $v_{S, A} \in I$. Then the corresponding vertex cover $U(x)$ is contained in U . Thus there always exists an optimum solution to I_2 that is of the form $U(x)$. Therefore, the reduction is exact.

It remains to compute the inapproximability factor via Theorem 4.4. We have

$$\rho_2 = \frac{1 - \varepsilon}{1 - (1 - 1/q - \varepsilon)} \geq q - \Theta(\varepsilon)$$

□

⁹In a hypergraph $H = (V, E)$ a set $I \subseteq V$ is said to be *independent* if no hyperedge of H is fully contained in I .

7 SDP-Hardness for INDEPENDENT SET

We saw in Section 4.2 how to obtain an LP-hardness for VERTEX COVER and INDEPENDENT SET, starting from an LP-hardness for the 1F-CSP problem. Restricting our starting CSP to have only *one free bit* is crucial for the VERTEX COVER problem, since each constraint is then represented by a *cloud* containing exactly two vertices in the resulting graph. In this case, an assignment satisfying *almost all* the constraints, corresponds to a vertex cover containing *slightly more than half* of the vertices (i.e., one vertex in almost all the clouds, and both vertices in the *unsatisfied* clouds), whereas if no assignment can simultaneously satisfy more than ε -fraction of the constraints, then any vertex cover should contain *almost all* the vertices. This extreme behaviour of the resulting graph is necessary to obtain a gap of 2 for the VERTEX COVER problem.

However, if we are only interested in the INDEPENDENT SET problem, any CSP with a sufficiently large gap between the soundness and completeness can yield the desired LP-Hardness, by virtue of the well-known FGLSS reduction [20]. Formally speaking, given reals $0 < s < c \leq 1$, and any CSP problem $\Pi(P, n, k)$, where n is the number of variables and P is a predicate of arity k , and knowing that no small linear program can provide a (c, s) -approximation for this CSP, then one can show that no small LP can as well approximate the INDEPENDENT SET problem within a factor of c/s . This can be simply done by tweaking the reduction of Section 4.2 in a way that the number of vertices in each cloud is equal to the number of satisfying assignments for the predicate. Hence dropping the *one free bit* requirement, and restricting ourselves to CSPs such that $c/s = 1/\varepsilon$ for arbitrarily small $\varepsilon := \varepsilon(k) > 0$, would yield the desired $\omega(1)$ LP-hardness for the INDEPENDENT SET problem.

Moreover, the reduction framework of [10] and our construction in Section 4.2 are agnostic to whether we are proving LP or SDP lower bounds, and hence having an analog of Theorem 4.8 in the SDP world would yield that any SDP of size less than $n^{\Omega(\log n / \log \log n)}$ has an integrality gap of $\omega(1)$ for the INDEPENDENT SET problem. In fact such SDP-hardness results for certain families of CSPs and hence an analog of Theorem 4.8 are known: if our starting CSP has a predicate that supports pairwise independence with a sufficiently large arity k , then the result of [5] by virtue of [37] gives us the desired SDP base hardness. By the argumentation from above we obtain:

Corollary 7.1. *For every $\varepsilon > 0$ and for infinitely many n , there exists a graph G with $|V(G)| = n$, such that no polynomial size SDP is a $(1/\varepsilon)$ -approximate SDP relaxation for INDEPENDENT SET(G).*

8 Discussion of related problems

We believe that our approach extends to many other related problems. As proved here, it applies to q -UNIFORM-VERTEX-COVER. Moreover, we would like to stress that our reduction is agnostic to whether it is used for LPs or SDPs and Lasserre gap instances for 1F-CSP, together with [38] and our reduction would provide SDP hardness of approximation for VERTEX COVER. This already holds for the INDEPENDENT SET problem as we saw in Section 7, since the starting CSP does not need to have only one free bit, as long as the gap between the soundness and completeness is sufficiently large.

Note that we are only able to establish hardness of approximations for the stable set problem within any constant factor, while assuming $P \neq NP$ one can establish hardness of approximation within $n^{1-\varepsilon}$. The reason for this gap is that the standard amplification techniques via graph products do not fall into the reduction framework in [10]. Also, there will be limits to amplification as established by the upper bounds in Section 5.

Finally, we would like to remark that our lower bounds on the size can be probably further strengthened, however, with our current reductions this would require a strengthened version of the results in [13].

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A Definition of Sherali-Adams for General Binary Linear Programs

For completeness, we give the general definition of the r -rounds SA tightening of a given LP, and then we show that for CSPs the obtained relaxation is equivalent to (2.1).

Consider the following Binary Linear Program for $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$:

$$\begin{aligned} \max \quad & \sum_{i=1}^n c_i x_i \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \{0, 1\}^n. \end{aligned}$$

By replacing the integrality constraint with $0 \leq x \leq 1$, we get an LP relaxation.

Sherali and Adams [50] proposed a systematic way for tightening such relaxations, by reformulating them in a higher dimensional space. Formally speaking, the r -rounds SA relaxation is obtained by multiplying each base inequality $\sum_{j=1}^n A_{ij}x_j \leq b_i$ by $\prod_{s \in S} x_s \prod_{t \in T} (1 - x_t)$ for all disjoint $S, T \subseteq [n]$ such that $|S \cup T| < r$. This gives the following set of polynomial inequalities for each such pair S and T :

$$\left(\sum_{j \in [n]} A_{ij} x_j \right) \prod_{s \in S} x_s \prod_{t \in T} (1 - x_t) \leq b_i \prod_{s \in S} x_s \prod_{t \in T} (1 - x_t) \quad \forall i \in [m],$$

$$0 \leq x_j \prod_{s \in S} x_s \prod_{t \in T} (1 - x_t) \leq 1 \quad \forall j \in [n].$$

These constraints are then linearized by first expanding (using $x_i^2 = x_i$, and thus $x_i(1 - x_i) = 0$), and then replacing each monomial $\prod_{i \in H} x_i$ by a new variable y_H , where $H \subseteq [n]$ is a set of size at most r . Naturally, we set $y_\emptyset := 1$. This gives us the following linear program, referred to as the r -rounds SA relaxation:

$$\begin{aligned} \max \quad & \sum_{i=1}^n c_i y_{\{i\}} \\ \text{s.t.} \quad & \sum_{H \subseteq T} (-1)^{|H|} \left(\sum_{j \in [n]} A_{ij} y_{H \cup S \cup \{j\}} \right) \leq b_i \sum_{H \subseteq T} (-1)^{|H|} y_{H \cup S} \quad \forall i \in [m], S, T, \\ & 0 \leq \sum_{H \subseteq T} (-1)^{|H|} y_{H \cup S \cup \{j\}} \leq 1 \quad \forall j \in [n], \forall S, T, \\ & y_\emptyset = 1 \end{aligned}$$

where in the first two constraint we take $S, T \subseteq [n]$ with $S \cap T = \emptyset$ and $|S \cup T| < r$.

One could go back to the original space by letting $x_i = y_{\{i\}}$ and projecting onto the x , however we will refrain from doing that, in order to be able to write objective functions that are not linear but degree- k polynomials, as is natural in the context of CSPs of arity k . Since we need to do k rounds of SA before even being able to write the objective function as a linear function, it makes more sense to work in higher dimensional space.

For CONSTRAINT SATISFACTION PROBLEMS, the canonical r -rounds SA relaxation is defined as follows. Consider any CSP defined over n variables $x_1, \dots, x_n \in [R]$, with m constraints $C =$

$\{C_1, \dots, C_m\}$ where the arity of each constraint is at most k . For each $j \in [n]$ and $u \in [R]$, we introduce a binary variable $x(j, u)$, meant to be the indicator of $x_j = u$. Using these variables, the set of feasible assignments can naturally be formulated as

$$\begin{aligned} \sum_{u \in [R]} x(j, u) &= 1 \quad \forall j \in [n], \\ x(j, u) &\in \{0, 1\} \quad \forall j \in [n], u \in [R]. \end{aligned}$$

If we relax the integrality constraints by, for each $j \in [n]$, $u \in [R]$, replacing $x(j, u) \in \{0, 1\}$ by $x(j, u) \geq 0$ (we omit the upper bounds of the form $x(j, u) \leq 1$ as they are already implied by the other constraints) then we obtain the following constraints for the r -rounds SA relaxation :

$$\sum_{H \subseteq T} (-1)^{|H|} \sum_{u \in [R]} y_{H \cup S \cup \{(j, u)\}} = \sum_{H \subseteq T} (-1)^{|H|} y_{H \cup S} \quad \forall j \in [n], S, T,$$

$$\sum_{H \subseteq T} (-1)^{|H|} y_{H \cup S \cup \{(j, u)\}} \geq 0 \quad \forall (j, u) \in [n] \times [R], S, T,$$

where we take $S, T \subseteq [n] \times [R]$ with $S \cap T = \emptyset$ and $|S \cup T| < r$.

To simplify the above description, we observe that we only need the constraints for which $T = \emptyset$.

Claim A.1. All the above constraints are implied by the subset of constraints for which $T = \emptyset$.

Proof. The equality constraints are easy to verify since $\sum_{u \in [R]} y_{S \cup \{(j, u)\}} = y_S$ for all $S \subseteq [n] \times [R]$ with $|S| < r$ implies

$$\sum_{S \subseteq H \subseteq S \cup T} (-1)^{|H \cap T|} \sum_{u \in [R]} y_{H \cup \{(j, u)\}} = \sum_{S \subseteq H \subseteq S \cup T} (-1)^{|S \cap T|} y_H.$$

Now consider the inequalities. If we let $T = \{(j_1, u_1), (j_2, u_2), \dots, (j_\ell, u_\ell)\}$ then by the above equalities

$$\begin{aligned} \sum_{H \subseteq T} (-1)^{|H|} y_{H \cup S \cup \{(j, u)\}} &= \sum_{H \subseteq T \setminus \{(j_1, u_1)\}} (-1)^{|H|} y_{H \cup S \cup \{(j, u)\}} - \sum_{H \subseteq T \setminus \{(j_1, u_1)\}} (-1)^{|H|} y_{H \cup S \cup \{(j, u), (j_1, u_1)\}} \\ &= \sum_{u'_1 \in [R]: u'_1 \neq u_1} \sum_{H \subseteq T \setminus \{(j_1, u_1)\}} (-1)^{|H|} y_{H \cup S \cup \{(j, u), (j_1, u'_1)\}} \\ &\vdots \\ &= \sum_{u'_t \in [R]: u'_t \neq u_t} \dots \sum_{u'_1 \in [R]: u'_1 \neq u_1} y_{S \cup \{(j, u), (j_1, u'_1), \dots, (j_t, u'_t)\}}. \end{aligned}$$

Hence, we have also that all the inequalities hold if they hold for those with $T = \emptyset$ and S such that $|S| < r$. \square

By the above claim, the constraints of the canonical r -rounds SA relaxation of the CSP can be simplified to:

$$\sum_{u \in [R]} y_{S \cup \{(j, u)\}} = y_S \quad \forall j \in [n], S \subseteq [n] \times [R] : |S| < r,$$

$$y_{S \cup \{(j,u)\}} \geq 0 \quad \forall (j,u) \in [n] \times [R], S \subseteq [n] \times [R] : |S| < r.$$

To see that this is equivalent to (2.1) observe first that $y_S = 0$ if $\{(j,u'), (j,u'')\} \subseteq S$. Indeed, by the partition constraint, we have

$$\sum_{u \in R} y_{\{(j,u'), (j,u'')\} \cup \{(j,u)\}} = y_{\{(j,u'), (j,u'')\}},$$

which implies the constraint $2y_{\{(j,u'), (j,u'')\}} \leq y_{\{(j,u'), (j,u'')\}}$. This in turn (together with the non-negativity) implies that $y_{\{(j,u'), (j,u'')\}} = 0$. Therefore, by again using the partition constraint, we have $y_S = 0$ whenever $\{(j,u'), (j,u'')\} \subseteq S$ and hence we can discard variables of this type. We now obtain the formulation (2.1) by using variables of type $X_{(\{j_1, \dots, j_t\}, \{u_1, \dots, u_t\})}$ instead of $y_{\{(j_1, u_1), (j_2, u_2), \dots, (j_t, u_t)\}}$. The objective function can be linearized, provided that the number of rounds is at least the arity of the CSP, that is $r \geq k$, so that variables for sets of cardinality k are available.

B Proof of Claim 3.6

Proof of Claim 3.6. Assume that Λ satisfies vw_1, \dots, vw_t simultaneously, i.e.,

$$\pi_{v,w_1}(\Lambda(w_1)) = \dots = \pi_{v,w_t}(\Lambda(w_t)) = \Lambda(v) \quad (\text{B.1})$$

and let $C_{x,S}$ and $C_{\bar{x},S}$ be the sub-cubes as in Figure 1. According to the new assignment, every variable $\langle w_i, z \rangle$ in the support of $C(v, \mathcal{W}_v, x, S)$ takes the value $z_{\Lambda(w_i)}$. Assume w.l.o.g. that $\langle w_i, z \rangle$ is such that $\pi_{v,w_i}^{-1}(z) \in C_{x,S}$, and let $y \in C_{x,S}$ satisfies $\pi_{v,w_i}(y) = z$. Then we get

$$z_{\Lambda(w_i)} = \pi_{v,w_i}(y)_{\Lambda(w_i)} = y_{\pi_{v,w_i}(\Lambda(w_i))} = y_{\Lambda(v)} \quad (\text{B.2})$$

where the last equality follows from (B.1). We know from the construction of the sub-cube $C_{x,S}$ that for all $j \notin S$ and for all $y \in C_{x,S}$, we have $y_j = x_j$. It then follows that if $\Lambda(v) \notin S$, equation B.2 yields that

$$z_{\Lambda(w_i)} = y_{\Lambda(v)} = x_{\Lambda(v)} \quad \forall \langle w_i, z \rangle \text{ s.t. } \pi_{v,w_i}^{-1}(z) \in C_{x,S}$$

Similarly, for the variables $\langle w_i, z \rangle$ with $\pi_{v,w_i}^{-1}(z) \in C_{\bar{x},S}$, we get that

$$z_{\Lambda(w_i)} = y_{\Lambda(v)} = \bar{x}_{\Lambda(v)} \quad \forall \langle w_i, z \rangle \text{ s.t. } \pi_{v,w_i}^{-1}(z) \in C_{\bar{x},S}$$

Thus far we proved that if Λ satisfies vw_1, \dots, vw_t simultaneously and $\Lambda(v) \notin S$, then $\psi(\Lambda)$ satisfies $C(v, \mathcal{W}_v, x, S)$. But we know by construction that $|S| = \varepsilon R$, and hence $\Lambda(v) \notin S$ with probability at least $1 - \varepsilon$. \square

C Proof of Claim 6.11

Proof of Claim 6.11. Assume that Λ satisfies vw_1, \dots, vw_t simultaneously, i.e.,

$$\pi_{v,w_1}(\Lambda(w_1)) = \dots = \pi_{v,w_t}(\Lambda(w_t)) = \Lambda(v) \quad (\text{C.1})$$

and let C_{x,S_ε} be the sub-cube as in Figure 2. For $z \in [q]^R$ with $\pi_{v,w_i}^{-1}(z) \in C_{x,S_\varepsilon}$, let $y \in [q]^R$ be such that $\pi_{v,w_i}(y) = z$. Recall that a constraint $C(v, \mathcal{W}_v, x, S_\varepsilon)$ looks as follows:

$$\langle w_i, z \oplus \tilde{z} \rangle \neq z_0 \quad \forall 1 \leq i \leq t, \forall z \text{ such that } \pi_{v,w_i}^{-1}(z) \in C_{x,S_\varepsilon} \quad (\text{C.2})$$

We now adopt the functions point of view, i.e., for a $w \in W$, the variables $\langle w, z \rangle$ for $z \in [q]^R$ with z_0 are the entries of the truth table of a function f_w , and according to the new assignment Λ , f_w is the *folded* dictatorship function of the label of $\Lambda(w)$.

So if we let $f := f_{w_i}$ for some $1 \leq i \leq t$, and $z := \langle w_i, z \rangle$, we get that

$$\langle w_i, z \oplus \tilde{z} \rangle \neq z_0 \iff f(z) \neq 0$$

and by our definition of the dictatorship function, the latter is zero iff $z_{\Lambda(w_i)} = 0$. But

$$z_{\Lambda(w_i)} = \pi_{v, w_i}(\mathbf{y})_{\Lambda(w_i)} = \mathbf{y}_{\pi_{v, w_i}(\Lambda(w_i))} = \mathbf{y}_{\Lambda(v)} \quad (\text{C.3})$$

where the last equality follows from (C.1). We know from the construction of the sub-cube C_{x, S_ε} that for all $j \notin S_\varepsilon$ and for all $y \in C_{x, S_\varepsilon}$, we have $y_j = x_j$. It then follows that if $\Lambda(v) \notin S_\varepsilon$, equation C.3 yields that

$$z_{\Lambda(w_i)} = \mathbf{y}_{\Lambda(v)} = x_{\Lambda(v)} \quad \forall \langle w_i, z \rangle \text{ s.t. } \pi_{v, w_i}^{-1}(z) \in C_{x, S_\varepsilon}$$

Moreover, given that x is chosen uniformly at random from $[q]^R$, we get that for any $i \in [R]$, $\mathbb{P}_{x \in [q]^R} [x_i = 0] = \frac{1}{q}$.

Thus far we proved that if Λ satisfies vw_1, \dots, vw_t simultaneously and $\Lambda(v) \notin S$, then $\psi(\Lambda)$ satisfies $C(v, \mathcal{W}_v, x, S)$ with probability $1 - \frac{1}{q}$. But we know by construction that $|S| = \varepsilon R$, and hence $\Lambda(v) \notin S$ with probability at least $1 - \varepsilon$. \square